Existence of torsional solitons in a beam model of suspension bridge

Vieri Benci*, Donato Fortunato†, Filippo Gazzola‡

Abstract

This paper studies the existence of solitons, namely stable solitary waves, in an idealized suspension bridge. The bridge is modeled as an unbounded degenerate plate, that is, a central beam with cross sections, and displays two degrees of freedom: the vertical displacement of the beam and the torsional angles of the cross sections. Under fairy general assumptions, we prove the existence of solitons. Under the additional assumption of large tension in the sustaining cables, we prove that these solitons have a nontrivial torsional component. This appears relevant for the security since several suspension bridges collapsed due to torsional oscillations.

Contents

AMS MSC2010: 74J35, 35C08, 35Q74, 35A15, 35B35.

Key words: Nonlinear beam equation, traveling solitary waves, hylomorphic solitons, variational methods.

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1 Introduction

A suspension bridge has a fairly complicated structure, involving several different components interacting in a nonlinear fashion. This generates serious difficulties when trying to set up a reliable mathematical model.

The nonlinear behavior of a bridge is confirmed by the appearance of traveling waves. Cone [1, IX-1], a chief engineer of the Golden Gate Bridge, observed some traveling waves during a windstorm on February 9, 1938: *I also observed that the suspended structure of the Bridge was undulating vertically in a wavelike motion of considerable amplitude ... the wave motion appeared to be a running wave similar to that made by cracking a whip.* One may also have a look at the video on the Volgograd Bridge [36]: although some people believe it is fake, it well describes what is meant by traveling waves.

For the Tacoma Narrows Bridge collapse [35] it is known that *the crucial event in the collapse is the sudden change from a vertical to a torsional mode of oscillation*, see [1, 34]. But the appearance of torsional oscillations is not an isolated event occurred only at the Tacoma Bridge: among others, we mention the collapses of the Brighton Chain Pier in 1836, of the Wheeling Suspension Bridge in 1854, of the Matukituki Suspension Footbridge in 1977. We refer to [19] for a detailed description of these collapses.

The purpose of the present paper is to emphasize the existence of traveling waves with a nontrivial torsional component within the following fish-bone model for "unbounded suspension bridges":

$$\begin{cases} u_{tt} + u_{xxxx} - \sigma u_{xx} + W'(u+\theta) + W'(u-\theta) = 0 \\ \theta_{tt} + \rho\theta_{xxxx} - \theta_{xx} + W'(u+\theta) - W'(u-\theta) = 0\,; \end{cases} \tag{1}$$

the derivation of this model and the meaning of the parameters therein ($\sigma, \rho \geq 0$) is described in detail in Section 2.2. We will prove that, under suitable assumptions, the system (1) admits solitons. Roughly speaking, a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some form of stability so that it has a particle-like behavior (see e.g. [7, 8] or [12]). Following [12] (or [7]), a soliton or solitary wave is called *hylomorphic* if its stability is due to a particular ratio between the *energy* E and the *hylenic charge* C which is another integral of motion. More precisely, a soliton $\mathbf{u}_0$ is hylomorphic if

$$E(\mathbf{u}_0) = \min\{E(\mathbf{u}) \mid C(\mathbf{u}) = C(\mathbf{u}_0)\}\,.$$

The physical meaning of C depends on the problem (in this case C is the *momentum*, see Section 4.1).

The first result of this paper is that, under mild assumptions, there exist hylomorphic solitons for the system (1). As mentioned earlier, torsional solitons are the relevant ones since they may lead to collapses and, possibly, one would like to prevent them. Our second main result states that if the cable tension σ is sufficiently large, then solitons have a torsional component, namely $\theta \not\equiv 0$. This means that stiffening the cables leads to a more dangerous situation. A similar conclusion was reached by Lazer-McKenna [24] who showed that if the Hooke constant of the hangers is sufficiently large, then some systems modeling suspension bridges admit a sign-changing periodic solution: these solutions appear because the equation operates in its nonlinear regime. Their conclusion [24, p.555] is that *strengthening a bridge can lead to its destruction.* In view of the above mentioned result, we reach the very same conclusion: *stiffening the cables of a suspension bridge can lead to its destruction.*

2 Description of the model

2.1 A brief history of previous models

The milestone contribution by Melan [31] views the deck of the bridge as a beam connected by hangers to a sustaining cable. Nonlinear infinite beams have been used in literature to highlight the existence of solitary waves, see [10, 30].

However, the beam model fails to display the two degrees of freedom of the deck: longitudinal and torsional. If one wishes to view torsional oscillations and to afford an explanation of these collapse, one

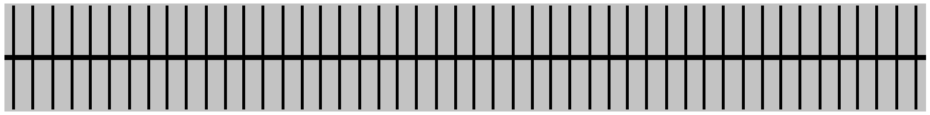

Figure 1: Unbounded fish-bone modeling the deck of an unbounded suspension bridge.

dimensional beam models should be discarded. On the other hand, modeling the deck of a suspension bridge as a plate is at the very beginning, see [2, 13, 17], and several important qualitative behaviors are still to be clarified. One is so led to introduce "intermediate models" which simplify the underlying equations but still allow to view the two main degrees of freedom of a suspension bridge.

McKenna [27] suggested to model the cross-section of the deck as a beam suspended by two lateral hangers. This gives rise to a nonlinear coupled system of two ODE's having as unknowns the vertical displacement of the barycenter of the beam and the torsional angle. His numerical results showed a sudden development of large torsional oscillations as soon as the hangers lose tension, that is, as soon as the restoring force becomes nonlinear. Further numerical results by Doole-Hogan [16] and McKenna-Tuama [28] show that a purely vertical periodic forcing may create a torsional response. This model was recently extended in [3] to a whole set of (coupled) cross sections and a system with a large number of oscillators was studied. By using some Poincaré maps, it was shown that when enough energy is present within the structure a resonance may occur, leading to an energy transfer between oscillators and to the sudden appearance of wide torsional oscillations. The numerical results in [3] show that the sudden transition between vertical and torsional oscillations is due to a *structural problem*. It is by now well-established [15, 18, 19, 22] that elastic structures made of metals and concrete, such as suspension bridges, behave nonlinearly and that the nonlinear structural behavior of the bridge seems to be responsible for the sudden transition between different kinds of oscillations.

2.2 The fish-bone model

In this section we describe in full detail the fish-bone model for the idealized suspension bridges. We are here concerned with the main span, namely the part of the deck between the towers, which has a rectangular shape with two long edges (of the order of 1km) and two shorter edges (of the order of 20m). The large discrepancy between these measures suggests to model the deck as a degenerate plate, that is, an infinite beam representing the midline of the deck with cross sections of given length $\ell > 0$ which are free to rotate around the beam. This model was called a *fish-bone* in [14], see Figure 1.

The grey part is the deck, the thick midline contains the barycenters of the cross sections and it is the line where the vertical displacement u will be computed. We emphasize that, contrary to some previous contributions in literature, the vertical axis is *oriented upwards*. The short thin orthogonal lines are virtual cross sections seen as rods that can rotate around their barycenter, the angle of rotation with respect to the horizontal position being denoted by α. For $x \in \mathbb{R}$ and $t > 0$, the equations describing this system read

$$\left\{\begin{array}{l} Mu_{tt} + EIu_{xxxx} - 2Hu_{xx} + W'(u + \ell\sin\alpha) + W'(u - \ell\sin\alpha) = 0 \\ \frac{M\ell^2}{3}\alpha_{tt} + EJ\ell^2\alpha_{xxxx} - \mu\ell^2\alpha_{xx} + \ell\cos\alpha\left(W'(u + \ell\sin\alpha) - W'(u - \ell\sin\alpha)\right) = 0, \end{array}\right. \tag{2}$$

where M is the mass density, $E > 0$ is the Young modulus, $EI > 0$ is the flexural rigidity of the beam, $EJ > 0$ is a geometric parameter of the cross-section, $\mu > 0$ is a constant depending on the shear modulus and the moment of inertia of the pure torsion, $H > 0$ is the tension in each sustaining cable, W' represents the restoring force of the hangers and also includes the action of gravity. In (2) we have not simplified by ℓ the second equation in order to emphasize all the terms. In a slightly different setting, involving mixed space-time fourth order derivatives, a *linear* version of (2) was suggested by Pittel-Yakubovich [33], see also [37, p.458, Chapter VI]. More recently, Moore [32] considered (2) with $H = EJ = 0$ and

$$W'(s) = k\left[\frac{Mg}{2k} - \left(\frac{Mg}{2k} - s\right)^+\right], \tag{3}$$

a nonlinearity which models hangers behaving as linear springs of elastic constant $k > 0$ if stretched but exert no restoring force if compressed; here g is gravity (warning: in [32] the vertical axis is oriented downwards). This nonlinearity, first suggested by McKenna-Walter [29], describes the possible slackening of the hangers which occurs for $s \geq \frac{Mg}{2k}$, that is, above the line where the hangers lose tension: slackening was observed during the Tacoma Bridge collapse, see [1, V-12]. Moore considers the case where the hangers do not slacken: then W' becomes *linear*, $W'(s) = ks$, the two equations in (2) decouple and there is obviously no interaction between vertical and torsional oscillations.

A nonlinear W' was considered in (2) by Holubová-Matas [21] who were able to prove well-posedness for a forced-damped version of (2). Also in [14] the well-posedness of an initial-boundary value problem for (2) is proved for a wide class of nonlinear forces W': both theoretical and numerical methods were used in order to display the instability, giving thereby an explanation for the origin of torsional oscillations.

Whence, the fish-bone model described by (2) is able to display a possible transition between vertical and torsional oscillations. In all the just quoted works it was assumed that $H = EJ = 0$. However, as pointed out in [4, 25], even if the hangers may indeed slacken and therefore display a nonlinear behavior, the most relevant contribution to instability is due to the sustaining cables. In this paper we will also take into account the behavior of the cables by inserting into (2) the parameter $H \geq 0$ representing their tension, as well as the geometric parameter $EJ \geq 0$. We are not interested in describing accurately the behavior of the bridge under large torsional oscillations and for small α the following approximations are legitimate

$$\cos\alpha \cong 1 \quad \text{and} \quad \sin\alpha \cong \alpha\,.$$

We set $\theta := \ell\alpha$ and this cancels the dependence of (2) on the width ℓ. After this change and after normalizing the constants, (2) may be rewritten as

$$\begin{cases} u_{tt} + u_{xxxx} - \sigma u_{xx} + W'(u+\theta) + W'(u-\theta) = 0 \\ \theta_{tt} + \rho\theta_{xxxx} - \theta_{xx} + W'(u+\theta) - W'(u-\theta) = 0 \end{cases} \tag{4}$$

where $u = u(t,x)$, $\theta = \theta(t,x)$, $W \in C^1(\mathbb{R})$, $\rho \geq 0$, and $\sigma \geq 0$ is the parameter which describes the tension of the cables, that is, the stiffness of the structure.

3 Hylomorphic solitons

3.1 An abstract definition of solitary waves and solitons

Solitary waves and solitons are particular *states* of a dynamical system described by one or more partial differential equations. We assume that the states of this system are described by one or more *fields* which mathematically are represented by functions

$$\mathbf{u} : \mathbb{R}^N \to V \tag{5}$$

where V is a vector space with norm $|\cdot|_V$ which is called the internal parameters space: for our problem we have $N = 1$ and $V = \mathbb{R}^4$. We assume the system to be deterministic; this means that it can be described as a dynamical system (X, γ) where X is the set of the states and $\gamma : \mathbb{R} \times X \to X$ is the time evolution map. The evolution of the system whose initial state is $\mathbf{u}_0(x) \in X$ is described by the function

$$\gamma_t \mathbf{u}_0(x). \tag{6}$$

We assume that the states of X have "finite energy" so that they decay at ∞ sufficiently fast.

We give a formal definition of solitary wave:

Definition 1 *A state $\mathbf{u}(x) \in X$ is called solitary wave if there is $\xi(t)$ such that*

$$\gamma_t \mathbf{u}(x) = \mathbf{u}(x - \xi(t)).$$

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well-known notions in the theory of dynamical systems.

Definition 2 *Let* (X, d) *be a metric space and let* (X, γ) *be a dynamical system. A set* $\Gamma \subset X$ *is called invariant if*

$$\gamma_t \mathbf{u} \in \Gamma \qquad \forall \mathbf{u} \in \Gamma,\ \forall t \in \mathbb{R}\ .$$

An invariant set $\Gamma \subset X$ *is called stable if*

$$\forall \varepsilon > 0, \exists \delta > 0, \Big(\mathbf{u} \in X, d(\mathbf{u}, \Gamma) \leq \delta\Big) \Rightarrow \Big(d(\gamma_t \mathbf{u}, \Gamma) \leq \varepsilon,\ \forall t \geq 0\Big)\,.$$

Let G be the group induced by the translations in $\mathbb{R}^N$, namely, for every $\tau \in \mathbb{R}^N$, the transformation $g_\tau \in G$ is defined as follows:

$$\left(g_\tau \mathbf{u}\right)(x) = \mathbf{u}\left(x - \tau\right). \tag{7}$$

Definition 3 *A functional* $J : X \to \mathbb{R}$ *is called* G*-invariant if*

$$\forall \tau \in \mathbb{R}^N,\ J\left(g_\tau \mathbf{u}\right) = J\left(\mathbf{u}\right).$$

A subset $\Gamma \subset X$ *is called* G*-invariant if*

$$\forall \mathbf{u} \in \Gamma,\ \forall \tau \in \mathbb{R}^N,\ g_\tau \mathbf{u} \in \Gamma.$$

Definition 4 *A closed* G*-invariant set* $\Gamma \subset X$ *is called* G*-compact if for any sequence* $\mathbf{u}_n(x)$ *in* Γ *there is a sequence* $\tau_n \in \mathbb{R}^N$*, such that* $\mathbf{u}_n(x - \tau_n)$ *has a converging subsequence.*

Now we are ready to give the definition of soliton:

Definition 5 *A solitary wave* $\mathbf{u}(x)$ *is called soliton if there is an invariant set* $\Gamma \subset X$ *such that*

$$(i)\ \mathbf{u}(x) \in \Gamma \qquad (ii)\ \Gamma \text{ is stable} \qquad (iii)\ \Gamma \text{ is } G\text{-compact.}$$

Usually, in the literature, the kind of stability described by the above definition is called *orbital stability.*

Remark 6 *The above definition needs some explanation. For simplicity, we assume that* Γ *is a manifold (actually, this is the generic case in many situations). Then (iii) implies that* Γ *is finite dimensional. Since* Γ *is invariant,* $\mathbf{u}_0 \in \Gamma \Rightarrow \gamma_t \mathbf{u}_0 \in \Gamma$ *for every time. Thus, since* Γ *is finite dimensional, the evolution of* $\mathbf{u}_0$ *is described by a finite number of parameters. Hence the dynamical system* (Γ, γ) *behaves as a point in a finite dimensional phase space. By the stability of* Γ*, a small perturbation of* $\mathbf{u}_0$ *remains close to* Γ*. However, in this case, its evolution depends on an infinite number of parameters. Therefore, this system appears as a finite dimensional system with a small perturbation. Since* $\dim(G) = N$, $\dim(\Gamma) \geq N$ *and hence, the "state" of a soliton is described by* N *parameters which define its position and, maybe, other parameters which define its "internal state".*

3.2 A general existence result of hylomorphic solitons

In some recent papers [6, 7, 9], the notion of *hylomorphic soliton* has been introduced and analyzed. The existence and the properties of hylomorphic solitons are guaranteed by the interplay between the *energy* E and an other integral of motion which, in the general case, is called *hylenic charge* and it will be denoted by C. More precisely:

Definition 7 *Assume that a dynamical system has two first integrals of motion* $E : X \to \mathbb{R}$ *and* $C : X \to \mathbb{R}$*. A soliton* $\mathbf{u}_0 \in X$ *is hylomorphic if* Γ *(as in Definition 5) has the following structure*

$$\Gamma = \Gamma\left(e_0, p_0\right) = \left\{\mathbf{u} \in X \mid E(\mathbf{u}) = e_0,\ C(\mathbf{u}) = p_0\right\}$$

where $e_0 = \min\left\{E(\mathbf{u}) \mid C(\mathbf{u}) = p_0\right\}$ *for some* $p_0 > 0$.

Clearly, for a given p_0 the minimum of E may not exist; moreover, even if it exists, it may happen that Γ does not satisfy (ii) or (iii) in Definition 5. In this section, we present an abstract theorem which guarantees the existence of hylomorphic solitons.

We set

$$\Lambda(\mathbf{u}) := \frac{E(\mathbf{u})}{C(\mathbf{u})}, \tag{8}$$

and

$$J_\beta(\mathbf{u}) = E(\mathbf{u}) + \beta\Lambda(\mathbf{u}) = \left(1 + \frac{\beta}{C(\mathbf{u})}\right) E(\mathbf{u}) \tag{9}$$

where β is a positive constant. Since E and C are constants of motion, also Λ and J_β are constants of motion; Λ will be called *hylenic ratio* and J_β will be called the *β-energy*. The importance of the functional J_β relies on the fact that, under very general assumptions (see below), it has a minimizer $\mathbf{u}_\beta$ in the set

$$X^+ = \{\mathbf{u} \in X |\ C(\mathbf{u}) > 0\}$$

and this minimizer is a hylomorphic soliton which satisfies the following equation:

$$E'(\mathbf{u}_\beta) = \lambda_\beta C'(\mathbf{u}_\beta) \tag{10}$$

where

$$\lambda_\beta = \frac{\beta\Lambda(\mathbf{u}_\beta)}{\beta + C(\mathbf{u}_\beta)}$$

is a Lagrange multiplier.

Next we seek sufficient conditions which guarantee the existence of hylomorphic solitons. To do this we need two definitions:

Definition 8 *A sequence $\mathbf{u}_n \in X$ is called G-vanishing sequence if for any sequence $\tau_n \subset G$ the sequence $\mathbf{u}_n(x - \tau_n)$ converges weakly to 0.*

Definition 9 *We say that a functional F on X has the splitting property if, given a sequence $\mathbf{u}_n = \mathbf{u} + \mathbf{w}_n \in X$ such that $\mathbf{w}_n$ converges weakly to 0, we have that*

$$F(\mathbf{u}_n) = F(\mathbf{u}) + F(\mathbf{w}_n) + o(1) \quad \text{as } n \to \infty.$$

Let us now list the assumptions on E and C:

- *(EC-0)* ***(Values at 0)*** $E(0) = C(0) = E'(0) = C'(0) = 0$.
- *(EC-1)* ***(Invariance)*** *$E(\mathbf{u})$ and $C(\mathbf{u})$ are G-invariant.*
- *(EC-2)* ***(Splitting property)*** *E and C satisfy the splitting property.*
- *(EC-3)* ***(Coercivity)*** *E satisfies*
  - (i) $E(\mathbf{u}) > 0 \quad \forall \mathbf{u} \neq 0$,
  - (ii) *if* $\|\mathbf{u}_n\| \to \infty$, *then* $E(\mathbf{u}_n) \to \infty$;
  - (iii) *if* $E(\mathbf{u}_n) \to 0$, *then* $\|\mathbf{u}_n\| \to 0$.

Finally, we state a sufficient condition which will be useful for our purposes.

Theorem 10 *Assume that E and C satisfy (EC-0), (EC-1), (EC-2), (EC-3), and that*

$$\inf_{\mathbf{u} \in X^+} \Lambda(\mathbf{v}) < \Lambda_0 \tag{11}$$

where $\Lambda_0 := \inf\{\lim\ \Lambda(\mathbf{u}_n) \mid \mathbf{u}_n \in X^+$ is a G-vanishing sequence$\}$. Then

$$\beta_0 := \inf\left\{\beta > 0 \mid \exists \mathbf{v} \in X^+ : \frac{1}{\beta}E(\mathbf{v}) + \Lambda(\mathbf{v}) < \Lambda_0 \right\} > 0$$

and for every $\beta \in (\beta_0, \infty)$ there exists a hylomorphic soliton $\mathbf{u}_\beta$ which is a minimizer of $J_\beta(\mathbf{u})$ (as defined in (9)) in X^+ and hence it satisfies (10).

The proof of this result may be found in [11, 12].

4 The main results

4.1 Statement of the main results

The system (4) has a variational structure, namely it is formed by the Euler-Lagrange equations with respect to the functional

$$S = \frac{1}{2}\int\int\left(u_t^2 - \sigma u_x^2 - u_{xx}^2\right)dx\ dt + \frac{1}{2}\int\int\left(\theta_t^2 - \theta_x^2 - \rho\theta_{xx}^2\right)dx\ dt$$
$$-\int\int\left[W(u+\theta) + W(u-\theta)\right]dx\ dt. \tag{12}$$

The system (4) can be rewritten as a Hamiltonian system as follows:

$$\begin{cases} u_t = \hat{u} \\ \theta_t = \hat{\theta} \\ \hat{u}_t = -u_{xxxx} + \sigma u_{xx} - W'(u+\theta) - W'(u-\theta) \\ \hat{\theta}_t = -\rho\theta_{xxxx} + \theta_{xx} - W'(u+\theta) + W'(u-\theta). \end{cases} \tag{13}$$

If $\rho > 0$, then the phase space is given by

$$X = \left[H^2(\mathbb{R})\right]^2 \times \left[L^2(\mathbb{R})\right]^2$$

and the generic point in X will be denoted by

$$\mathbf{u} = \left(u, \theta, \hat{u}, \hat{\theta}\right).$$

Notice that we denote by $\hat{u}$ and $\hat{\theta}$ the conjugate variables of u and θ respectively. We equip X with the following norm:

$$\|\mathbf{u}\|^2 = \int\left(\hat{u}^2 + \hat{\theta}^2 + u_{xx}^2 + \sigma u_x^2 + \rho\theta_{xx}^2 + \theta_x^2 + 2u^2 + 2\theta^2\right)dx \tag{14}$$

which is a norm equivalent to the standard $\left[H^2(\mathbb{R})\right]^2 \times \left[L^2(\mathbb{R})\right]^2$-norm.

If $\rho = 0$, then the phase space is given by

$$X = H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times \left[L^2(\mathbb{R})\right]^2$$

and the norm is again (14) with $\rho = 0$.

The Lagrangian relative to the action (12) is

$$\mathcal{L} = \frac{1}{2}\left(u_t^2 - u_{xx}^2 - \sigma u_x^2\right) + \frac{1}{2}\left(\theta_t^2 - \rho\theta_{xx}^2 - \theta_x^2\right) - W(u+\theta) - W(u-\theta). \tag{15}$$

This Lagrangian is invariant for time and space translations. Then, by Noether's Theorem (see e.g. [8, 20]), the energy E and the momentum C defined by

$$E = \int\left(\frac{\partial\mathcal{L}}{\partial u_t}u_t + \frac{\partial\mathcal{L}}{\partial\theta_t}\theta_t - \mathcal{L}\right)dx$$
$$= \frac{1}{2}\int\left(u_t^2 + u_{xx}^2 + \sigma u_x^2 + \theta_t^2 + \rho\theta_{xx}^2 + \theta_x^2\right)dx + \int\left[W(u+\theta) + W(u-\theta)\right]dx$$

$$C = \int\left(\frac{\partial\mathcal{L}}{\partial u_t}u_x + \frac{\partial\mathcal{L}}{\partial\theta_t}\theta_x\right)dx = \int\left(u_t u_x + \theta_t\theta_x\right)\ dx$$

are constant along the solutions of (4). Our purpose is to apply the abstract theory of Section 3 to these functionals, where the momentum $C(\mathbf{u})$ plays the role of the hylenic charge. On the function W we make the following assumptions:

- (W-i) **(Positivity)** $W \in C^2(\mathbb{R})$, $W(s) > 0$ $\forall s \neq 0$, and $\exists \eta > 0$ such that $W(s) \geq \eta$ for $|s| \geq 1$.
- (W-ii) **(Nondegeneracy at 0)** $W(s) = \frac{1}{2}s^2 + N(s)$ with $N(s) = o(s^2)$ as $s \to 0$.
- (W-iii) **(One-sided subquadratic growth)** $\exists M > 0,\ \exists \alpha \in [0, 2)$,

$$W(s) \leq M\, s^\alpha \qquad \forall s \geq 0.$$

Note that assumption (W-ii) implies that $W'(0) = 0$ and $W''(0) = 1$; moreover, $W(u+\theta)+W(u-\theta) = u^2 + \theta^2 + N(u+\theta) + N(u-\theta)$. Then the energy and the momentum, as functionals defined on X, take the following form

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2}\int \left(\hat{u}^2 + u_{xx}^2 + \sigma u_x^2 + \hat{\theta}^2 + \rho\theta_{xx}^2 + \theta_x^2\right) dx + \int \left[W(u+\theta) + W(u-\theta)\right] dx \\ &= \frac{1}{2}\|\mathbf{u}\|^2 + \int \left[N(u+\theta) + N(u-\theta)\right] dx, \end{aligned} \tag{16}$$

$$C(\mathbf{u}) = \int \left(\hat{u}u_x + \hat{\theta}\theta_x\right)\ dx.$$

Also assumption (W-iii) deserves some attention: it requires a *strictly sublinear growth* for W' at $+\infty$. In particular, except for the C^2-smoothness, all these assumptions are satisfied by (see (3)):

$$W(s) = \begin{cases} \frac{1}{2}s^2 & for\ \ s \leq 1 \\ s - \frac{1}{2} & for\ \ s \geq 1 \end{cases} \tag{17}$$

The system (4) with the function $W(s)$ as in (17) and $\theta = 0$ has been proposed as model for a suspension bridge; see [23, 24, 29]. Once more, we recall that in these papers the vertical axis is oriented downwards. Let us also mention that the forms

$$W(s) = s - 1 + e^{-s} \quad \text{and} \quad W(s) = s + \frac{s^2}{2} - \frac{s\sqrt{1+s^2}}{2} - \frac{\log(s + \sqrt{1+s^2})}{2}$$

have been considered, respectively, in [30] and [26] as possible smooth alternative choices for the potential; these lead, respectively, to $W'(s) = 1 - e^{-s}$ and $W'(s) = 1 + s - \sqrt{1+s^2}$ for the nonlinear restoring force. Also these functions satisfy (W-i), (W-ii), and (W-iii); however, they fail to satisfy (22) below. For this reason, we suggest here a family of smooth functions. For all $a \in [0, 1]$, we consider

$$W(s) = \begin{cases} \frac{a}{2}s^2 + (1-a)(\sqrt{1+s^2} - 1) & for\ \ s \leq 0 \\ \sqrt{1+s^2} - 1 & for\ \ s \geq 0\,. \end{cases} \tag{18}$$

These are examples of smooth (C^2) functions satisfying (W-i), (W-ii), and (W-iii); in the limit case $a = 1$, the force W' is linear for $s \leq 0$.

We have the following existence result.

Theorem 11 *Assume that (W-i),(W-ii),(W-iii) hold and that*

$$\sigma, \rho \geq 0\,,\ \ with\ \sigma < 1 + 4\sqrt{2\rho}\ \ if\ \sigma \geq 1\,. \tag{19}$$

Then there exists an open interval $I \subset \mathbb{R}_+$ such that, for every $\beta \in I$, there is an hylomorphic soliton $\mathbf{u}_\beta$ for the dynamical system (13) which is a minimizer of J_β in X^+ (see (9)) and hence it satisfies (10).

Theorem 11 holds under the assumption (19) for the couple (σ, ρ). This assumption is equivalent to $\rho > (\sigma - 1)^2/32$ if $\sigma \geq 1$ and this region is represented in Figure 2.

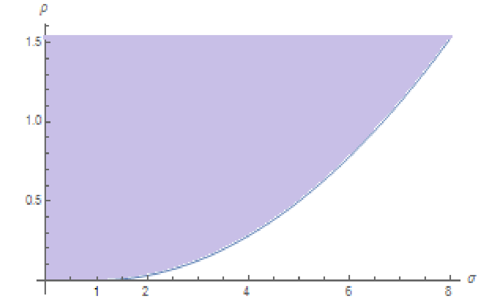

Figure 2: Region of validity in the (σ, ρ)-plane for Theorem 11.

Moreover, we may characterize hylomorphic solitons by means of the following statement.

Theorem 12 *Let* $\mathbf{u}_\beta = \big(u_\beta, \theta_\beta, \hat{u}_\beta, \hat{\theta}_\beta\big)$ *be a soliton as in Theorem 11. Then the solution of system (4) with initial data* $\big(u_\beta, \theta_\beta, \hat{u}_\beta, \hat{\theta}_\beta\big)$ *has the form:*

$$u(t,x) = u_\beta(x-\lambda t),\ \ \theta(t,x) = \theta_\beta(x-\lambda t)\,.$$

Moreover, (u_β, θ_β) *is a solution of the following ODE system*

$$u_\beta'''' + (\lambda^2 - \sigma)u_\beta'' + W'(u_\beta - \theta_\beta) + W'(u_\beta + \theta_\beta) = 0 \tag{20}$$

$$\theta_\beta'''' + (\lambda^2 - 1)\theta_\beta'' + W'(u_\beta + \theta_\beta) - W'(u_\beta - \theta_\beta) = 0 \tag{21}$$

and $\lambda > 0$ *is a constant which depends on* β*, see (42) below.*

The proofs of Theorem 11 and of Theorem 12 will be given in the next section.

We wish now to describe the *shape* of solitons. A **longitudinal soliton** is a soliton with $u \neq 0$ whereas a **torsional soliton** is a soliton with $\theta \neq 0$. Their shape is sketched in the pictures below. In Figure 3 we see two kinds of solitary waves: on the left a purely longitudinal soliton ($u \neq 0$, $\theta = 0$), on the right a purely torsional soliton ($u = 0$, $\theta \neq 0$).

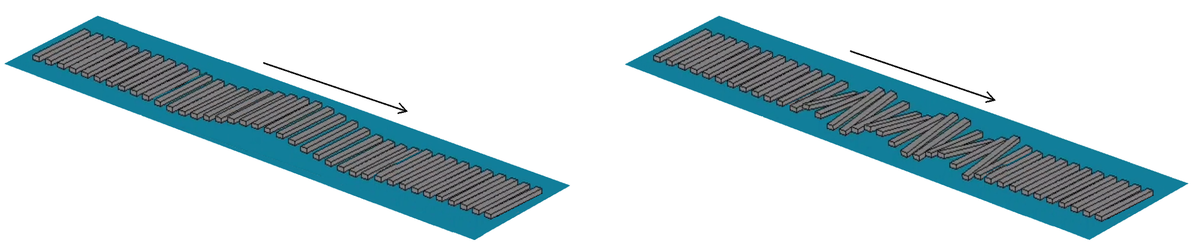

Figure 3: Representation of purely longitudinal (left) and purely torsional (right) solitons.

As already mentioned in the Introduction, the most dangerous situation for the fish-bone structure is the appearance of a torsional component ($\theta \neq 0$). Whether conditions (W-i), (W-ii), and (W-iii) are sufficient to guarantee the existence of torsional solitons is an open problem. In this paper we strengthen these assumptions by requiring also the following inequalities:

$$\begin{gathered}\rho \leq 1 \leq \sigma < 1 + 4\sqrt{2\rho} \qquad \text{and} \qquad W''(s)s^2 \leq W'(s)s \quad \forall s \neq 0\,, \\ \text{and at least one of the three non-strict inequalities is strict.}\end{gathered} \tag{22}$$

An example of function W satisfying (W-i), (W-ii), (W-iii), and (22) is given by (18). If W is given by (17) then condition (22) holds a.e. since $W''(1)$ is not defined: however, Theorem 13 below remains true also in this case with minor changes in the proof.

Then Theorems 11 and 12 can be refined by the following statement.

Theorem 13 *Assume that (W-i), (W-ii), (W-iii) and (22) hold, then every hylomorphic soliton* $\mathbf{u}_\beta$ *($\beta \in I \subset \mathbb{R}_+$) for the dynamical system (13) is a torsional soliton.*

Theorem 13 holds under the assumption (22) for the couple (σ, ρ) and this region is represented in Figure 4.

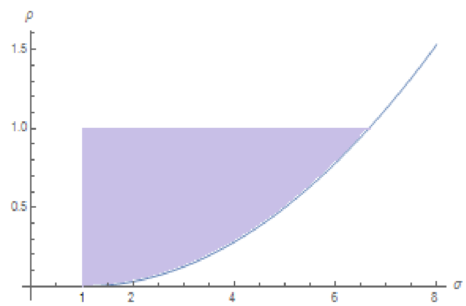

Figure 4: Region of validity in the (σ, ρ)-plane for Theorem 13.

The proof of Theorem 13 is in the next section. We point out that Theorem 13 does not exclude the existence of solitary waves with possibly zero torsional component. And, indeed, these solitary waves do exist; to see this, one can argue directly on system (13) by imposing that the θ-component is identically 0. On the other hand, Theorem 13 states that if (22) holds then these solitons are not hylomorphic and that the only hylomorphic solitons are torsional (with $\theta \neq 0$). Since hylomorphy means stability due to a particular ratio between the energy and the hylenic charge, this shows that traveling waves having this form of stability in an infinite beam with cross sections and with large tension in the cables ($\sigma \geq 1$) necessarily display a torsional component.

Even if less interesting from the engineering point of view, a natural mathematical question is to find out if the solitons have a longitudinal component ($u \neq 0$), that is, if they are longitudinal solitons as previously defined. Also in this case we need a further assumption, the counterpart of (22):

$$\begin{gathered}\sigma \leq 1 \leq \rho \qquad \text{and} \qquad W''(s)s^2 \leq W'(s)s \quad \forall s \neq 0\,, \\ \text{and at least one of the three non-strict inequalities is strict.}\end{gathered} \tag{23}$$

Then, in next section, we prove:

Theorem 14 *Assume that (W-i), (W-ii), (W-iii) and (23) hold, then every hylomorphic soliton* $\mathbf{u}_\beta$ *($\beta \in I \subset \mathbb{R}$) for the dynamical system (13) is a longitudinal soliton.*

Theorem 14 holds under the assumption (23) for the couple (σ, ρ) and this region is represented in Figure 5.

Finally, by combining Theorems 13 and 14 we obtain a somehow trivial result concerning the "symmetric case":

Corollary 15 *Assume that (W-i), (W-ii), (W-iii), and*

$$\sigma = \rho = 1 \quad , \qquad W''(s)s^2 < W'(s)s \quad \forall s \neq 0\,.$$

Then every hylomorphic soliton $\mathbf{u}_\beta$ *($\beta \in I \subset \mathbb{R}$) for the dynamical system (13) is both longitudinal and torsional.*

Examples of functions W satisfying the assumptions of Corollary 15 are given in (18) with $a \in [0, 1)$, $a = 1$ must be excluded here. Corollary 15 holds under the restrictive assumption that $(\sigma, \rho) = (1, 1)$. In Figure 6 we represent this point as intersection of the two regions defined in Theorems 13 and 14, see Figures 4 and 5.

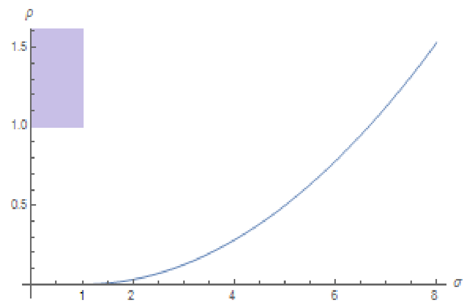

Figure 5: Region of validity in the (σ,ρ)-plane for Theorem 14.

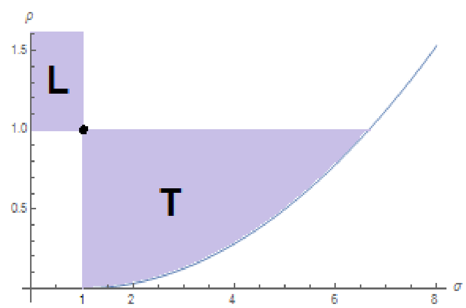

Figure 6: Intersection of the two regions described by Figures 4 and 5.

There exists $\overline{\sigma}>0$ such that if $\sigma\le-\overline{\sigma}$ then the energy functional E in (16) fails to satisfy *(EC-3)*; in this case, Theorem 11 is no longer true. Moreover, it is not clear whether the restriction $\sigma\ge1$ in (22) is necessary for the existence of a torsional soliton. However, it is not difficult to prove (just taking $\theta=0$ and following the proof of [10]) that if $-\overline{\sigma}<\sigma<1$, then the dynamical system (13) admits nontorsional solitary waves (with $\theta\equiv0$) which are local minimizers of the energy E and, hence, stable solitons. Finally, notice that (22) implicitly requires that $\rho>0$: it is a challenging problem to prove (or disprove) Theorem 13 when $\rho=0$.

Finally, let us give a look at the *quantitative aspect* of the above results since, going from (2) to (4), we ruled out several constants. Back to the original system (2), let us quote the values of the parameters of the collapsed Tacoma Narrows Bridge taken from [1]:

$$M=2790\,kg/m,\ E=2\cdot10^{11}\,N/m^2,\ I=0.15\,m^4,\ H=58.3\cdot10^6\,N,\ \ell=6\,m,\ \mu=14.5\,kN,$$

while J is not computed in [1] but it is usually very small in common bridges, see [22]. Part of these constants can be absorbed into the unknown thanks to suitable time and space scaling. However, in the studied system (4) the constant ρ is usually small as in (22). Therefore, the most relevant result is Theorem 13 and torsional solitons may appear if the cable tension σ is too large.

4.2 Proof of the main results

In order to prove Theorem 11, we will show that all the assumptions of Theorem 10 are satisfied with

$$X=\left[H^2(\mathbb{R})\right]^2\times\left[L^2(\mathbb{R})\right]^2\quad(\rho>0)\,,\qquad X=H^2(\mathbb{R})\times H^1(\mathbb{R})\times\left[L^2(\mathbb{R})\right]^2\quad(\rho=0)\,,$$

$$E(\mathbf{u})=\frac{1}{2}\left\|\mathbf{u}\right\|^2+\int\left[N(u+\theta)+N(u-\theta)\right]dx\,,\qquad C\left(\mathbf{u}\right)=\int\left(\hat{u}u_x+\hat{\theta}\theta_x\right)\,dx$$

and G is the group of translations in x. We first prove the splitting property.

Lemma 16 *Assumption (EC-2) holds, that is, E and C satisfy the splitting property, see Definition 9.*

Proof. The proof can be found in [12] Lemma 125, pag.210 (see also [10], Lemma 21). Actually in the mentioned references the splitting property refers to a single equation while here it refers to a system. However no relevant change is required. $\square$

Then we prove a crucial bound for the norm of $\mathbf{u}$ in terms of its energy.

Lemma 17 *There exists a continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(0) = 0$ and*

$$\|\mathbf{u}\| \leq h(E(\mathbf{u})) \qquad \forall \mathbf{u} \in X.$$

Proof. From (16) and assumption (W-i) we infer that

$$\frac{1}{2}\|\mathbf{u}\|^2 = E(\mathbf{u}) - \int [N(u+\theta) + N(u-\theta)]\, dx \leq E(\mathbf{u}) + \int \left(u^2 + \theta^2\right) dx. \tag{24}$$

Then it is sufficient to prove that

$$\int \left(u^2 + \theta^2\right) dx \leq h_1(E(\mathbf{u}))$$

where $h_1 : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that $h_1(0) = 0$. We set

$$\begin{aligned}
\Omega &= \left\{x \in \mathbb{R} \mid u(x)^2 + \theta(x)^2 < 2\right\}, \\
\Omega_u^+ &= \{x \in \mathbb{R} \mid u(x) \geq 1\}, \quad \Omega_u^- = \{x \in \mathbb{R} \mid u(x) \leq -1\}, \quad \Omega_u = \Omega_u^+ \cup \Omega_u^-, \\
\Omega_\theta^+ &= \{x \in \mathbb{R} \mid \theta(x) \geq 1\}, \quad \Omega_\theta^- = \{x \in \mathbb{R} \mid \theta(x) \leq -1\}, \quad \Omega_\theta = \Omega_\theta^+ \cup \Omega_\theta^-,
\end{aligned}$$

so that $\Omega \cup \Omega_u \cup \Omega_\theta = \mathbb{R}$. By (W-i) we have that, for any $x \in \Omega_u$,

$$W(u+\theta) + W(u-\theta) \geq \eta$$

and hence

$$E(\mathbf{u}) \geq \int_{\Omega_u} [W(u+\theta) + W(u-\theta)]\, dx \geq \eta \cdot \left(\left|\Omega_u^+\right| + \left|\Omega_u^-\right|\right);$$

therefore

$$\left|\Omega_u^+\right| \leq \frac{E(\mathbf{u})}{\eta}, \quad \left|\Omega_u^-\right| \leq \frac{E(\mathbf{u})}{\eta}, \quad \text{and} \quad |\Omega_u| \leq \frac{E(\mathbf{u})}{\eta}. \tag{25}$$

Set $v = u - 1$, then, since $v = 0$ on $\partial\Omega_u^+$, by the Poincaré inequality,

$$\int_{\Omega_u^+} v^2 dx \leq \left|\Omega_u^+\right|^2 \int_{\Omega_u^+} v_x^2 dx \leq \frac{E(\mathbf{u})^2}{\eta^2} \int_{\Omega_u^+} v_x^2 dx. \tag{26}$$

On the other hand, an integration by parts and Hölder's inequality yield

$$\int_{\Omega_u^+} v_x^2 dx = -\int_{\Omega_u^+} v\, v_{xx} dx \leq \|v\|_{L^2(\Omega_u^+)} \|v_{xx}\|_{L^2(\Omega_u^+)} \leq \|v\|_{L^2(\Omega_u^+)} \sqrt{2E(\mathbf{u})}. \tag{27}$$

By combining (26) and (27) we obtain

$$\|v\|_{L^2(\Omega_u^+)}^2 \leq \frac{E(\mathbf{u})^2}{\eta^2} \int_{\Omega_u^+} v_x^2 dx \leq \frac{E(\mathbf{u})^2}{\eta^2} \|v\|_{L^2(\Omega_u^+)} \sqrt{2E(\mathbf{u})},$$

$$\|v\|_{L^2(\Omega_u^+)} \leq \frac{\sqrt{2}\, E(\mathbf{u})^{5/2}}{\eta^2}.$$

Then, since $v = u - 1$, we get

$$\|u - 1\|_{L^2(\Omega_u^+)} \leq \frac{\sqrt{2}\, E(\mathbf{u})^{5/2}}{\eta^2}$$

In turn, since

$$\|u-1\|^2_{L^2(\Omega_u^+)} = \|u\|^2_{L^2(\Omega_u^+)} - 2\int\limits_{\Omega_u^+} u\ dx + \left|\Omega_u^+\right|\ ,$$

we infer that

$$\|u\|^2_{L^2(\Omega_u^+)} - 2\int\limits_{\Omega_u^+} u\ dx + \left|\Omega_u^+\right| \le \frac{2\,E\left(\mathbf{u}\right)^5}{\eta^4}$$

and hence, by (25),

$$\begin{aligned}\|u\|^2_{L^2(\Omega_u^+)} &\le 2\int\limits_{\Omega_u^+} u\ dx - \left|\Omega_u^+\right| + \frac{2\,E\left(\mathbf{u}\right)^5}{\eta^4} \le 2\,\|u\|_{L^2(\Omega_u^+)}\left|\Omega_u^+\right|^{1/2} + \frac{2\,E\left(\mathbf{u}\right)^5}{\eta^4}\\ &\le 2\,\|u\|_{L^2(\Omega_u^+)}\left(\frac{E\left(\mathbf{u}\right)}{\eta}\right)^{1/2} + \frac{2\,E\left(\mathbf{u}\right)^5}{\eta^4}.\end{aligned}$$

We finally infer that

$$\|u\|^2_{L^2(\Omega_u^+)} \le \sqrt{\frac{E\left(\mathbf{u}\right)}{\eta}} + \sqrt{\frac{E\left(\mathbf{u}\right)}{\eta} + \frac{2\,E\left(\mathbf{u}\right)^5}{\eta^4}} =: h_2\left(E\left(\mathbf{u}\right)\right)$$

where h_2 is continuous and satisfies $h_2(0) = 0$. With analogous arguments, one may obtain a similar estimate on Ω_u^- and therefore

$$\int_{\Omega_u} u^2 dx \le h_3\left(E\left(\mathbf{u}\right)\right) \tag{28}$$

for some continuous function h_3 satisfying $h_3(0) = 0$. Furthermore, with the same arguments one may obtain the θ-counterparts of (25) and (28):

$$\left|\Omega_\theta^+\right| \le \frac{E\left(\mathbf{u}\right)}{\eta} \quad \text{and} \quad \left|\Omega_\theta^-\right| \le \frac{E\left(\mathbf{u}\right)}{\eta}\ , \tag{29}$$

$$\int_{\Omega_\theta} \theta^2 dx \le h_4\left(E\left(\mathbf{u}\right)\right) \tag{30}$$

for some continuous function h_4 satisfying $h_4(0) = 0$.

Since $|\theta| \le 1$ in $\Omega_u \setminus \Omega_\theta$, we obtain

$$\begin{aligned}\int_{\Omega_u}(u^2+\theta^2)dx &= \int_{\Omega_u\cap\Omega_\theta}(u^2+\theta^2)dx + \int_{\Omega_u\setminus\Omega_\theta}(u^2+\theta^2)dx\\ \text{by (25), (28), (30)}\quad &\le h_3\left(E\left(\mathbf{u}\right)\right) + h_4\left(E\left(\mathbf{u}\right)\right) + h_3\left(E\left(\mathbf{u}\right)\right) + \frac{E\left(\mathbf{u}\right)}{\eta}\\ &= h_5\left(E\left(\mathbf{u}\right)\right)\end{aligned} \tag{31}$$

for some continuous function h_5 satisfying $h_5(0) = 0$. Similarly, by using (28), (29), (30), and the fact that $|u| \le 1$ in $\Omega_\theta \setminus \Omega_u$, we infer that

$$\int_{\Omega_\theta}(u^2+\theta^2)dx \le h_6\left(E\left(\mathbf{u}\right)\right) \tag{32}$$

for some continuous function h_6 satisfying $h_6(0) = 0$.

From (W-i) and (W-ii) we deduce that

$$a := \inf_{|s|<2}\frac{W(s)}{s^2} > 0.$$

Hence, if $x \in \Omega$, then $W(u+\theta)+W(u-\theta) \geq a(u+\theta)^2 + a(u-\theta)^2 = 2a\big(u^2+\theta^2\big)$ and therefore

$$\int_\Omega \left(u^2+\theta^2\right) dx \leq \frac{1}{2a}\int_\Omega \left[W(u+\theta)+W(u-\theta)\right] dx \leq \frac{E(\mathbf{u})}{2a}. \tag{33}$$

We may now conclude. From (24), (31), (32), and (33) we infer that

$$\begin{aligned}
\frac{1}{2}\|\mathbf{u}\|^2 &\leq E(\mathbf{u}) + \int \left(u^2+\theta^2\right) dx \\
&\leq E(\mathbf{u}) + \int_\Omega \left(u^2+\theta^2\right) dx + \int_{\Omega_u} \left(u^2+\theta^2\right) dx + \int_{\Omega_\theta} \left(u^2+\theta^2\right) dx \\
&\leq E(\mathbf{u}) + \frac{E(\mathbf{u})}{2a} + h_5\left(E(\mathbf{u})\right) + h_6\left(E(\mathbf{u})\right)
\end{aligned}$$

and the statement follows by taking $h(s) = \big(2+1/a\big)s + 2h_5(s) + 2h_6(s)$. □

We now turn our attention on the behavior of G-vanishing sequences. The following result states that this is a strong form of vanishing.

Lemma 18 *Let $\mathbf{u}_n = \big(u_n, \theta_n, \hat{u}_n, \hat{\theta}_n\big) \in X$ be a G-vanishing sequence; then u_n and θ_n converge uniformly to 0 in $\mathbb{R}$.*

Proof: Let x_n be one of the maximum points of $|u_n|$; since $u_n(x+x_n)$ converges weakly to 0 in $H^1(\mathbb{R})$, we have that it converges strongly to 0 in $L^\infty(-1,1)$ and hence $|u_n(x+x_n)| \to 0$ for every $x \in (-1,1)$; so $\|u_n\|_{L^\infty} = |u_n(x_n)| \to 0$. The proof for θ_n is similar. □

The next lemma is needed to verify the *hylomorphy condition* (11).

Lemma 19 *Let Λ_0 be as in (11), that is,*

$$\Lambda_0 := \ \inf\left\{\lim\ \Lambda(\mathbf{u}_n) \mid \mathbf{u}_n \in X^+ \text{ is a } G\text{-vanishing sequence}\right\}. \tag{34}$$

Then $\Lambda_0 \geq \Upsilon := \min\{\sigma + 2\sqrt{2}\,,\, 1+2\sqrt{2\rho}\}^{1/2}$.

Proof: Let $\mathbf{u}_n = \big(u_n, \theta_n, \hat{u}_n, \hat{\theta}_n\big) \in X^+$ be a G-vanishing sequence. By Lemma 18 we know that u_n and θ_n converge uniformly to 0. By (W-ii) we have that

$$|N(s)| \leq h(s)s^2$$

with $h(s) \to 0$ for $s \to 0$. These two facts yield

$$\left|\int N(u_n+\theta_n)dx\right| \leq \int h(u_n+\theta_n)(u_n+\theta_n)^2 dx \leq 2\left\|h(u_n+\theta_n)\right\|_{L^\infty}\left[\|u_n\|_{L^2}^2 + \|\theta_n\|_{L^2}^2\right]$$

and hence $\int N(u_n+\theta_n)dx \to 0$. Similarly, $\int N(u_n-\theta_n)dx \to 0$. We have so proved that

$$\int \left[N(u_n+\theta_n)+N(u_n-\theta_n)\right] dx \to 0 \quad \text{as } n \to \infty \tag{35}$$

for any G-vanishing sequence $\mathbf{u}_n = \big(u_n, \theta_n, \hat{u}_n, \hat{\theta}_n\big) \in X$.

We first prove the statement in the case where $\rho > 0$. By definition of Λ_0, see (34), for any $\varepsilon > 0$, there exists a G-vanishing sequence $\mathbf{u}_n \in X^+$ such that

$$\begin{aligned}
\Lambda_0 + \varepsilon &\geq \lim_n\ \Lambda(\mathbf{u}_n) \\
&= \lim_n\ \frac{\frac{1}{2}\|\mathbf{u}_n\|^2 + \int\left[N(u_n+\theta_n)+N(u_n-\theta_n)\right]dx}{\int\left(\hat{u}_n u_{n,x} + \hat{\theta}_n\theta_{n,x}\right)dx} \qquad \text{(by (8) and (16))} \\
&= \lim_n\ \frac{\|\mathbf{u}_n\|^2}{2\int\left(\hat{u}_n u_{n,x} + \hat{\theta}_n\theta_{n,x}\right)dx}. \qquad \text{(by (35))}
\end{aligned}$$

To continue the estimate we need to apply several times the Young inequality

$$2|ab| \leq \frac{1}{\delta}(a^2 + \delta^2 b^2) \qquad \forall a, b \in \mathbb{R}\ , \quad \forall \delta > 0\,. \tag{36}$$

In particular, an integration by parts and (36) yield

$$\|v'\|_{L^2}^2 = \int (v')^2 dx = \left|\int v'' v dx\right| \leq \int |v'' v| \leq \frac{1}{2\sqrt{2\eta}} \int \left(\eta (v'')^2 + 2v^2\right) \qquad \forall v \in H^2(\mathbb{R})\,. \tag{37}$$

Then we obtain

$$\begin{aligned}
\Lambda_0 + \varepsilon &\geq \lim_n \frac{\int \left(\hat{u}_n^2 + \hat{\theta}_n^2 + u_{n,xx}^2 + \sigma u_{n,x}^2 + 2u_n^2 + \rho\theta_{n,xx}^2 + \theta_{n,x}^2 + 2\theta_n^2\right) dx}{2\int \left(|\hat{u}_n u_{n,x}| + |\hat{\theta}_n \theta_{n,x}|\right) dx} && \text{by (14)} \\
&\geq \lim_n \frac{\int \left(\hat{u}_n^2 + \hat{\theta}_n^2 + (\sigma + 2\sqrt{2}) u_{n,x}^2 + (1 + 2\sqrt{2\rho})\theta_{n,x}^2\right) dx}{2\int \left(|\hat{u}_n u_{n,x}| + |\hat{\theta}_n \theta_{n,x}|\right) dx} && \text{by applying (37) twice} \\
&\geq \lim_n \frac{\int \left(\hat{u}_n^2 + \hat{\theta}_n^2 + \Upsilon^2 u_{n,x}^2 + \Upsilon^2 \theta_{n,x}^2\right) dx}{\frac{1}{\Upsilon}\int \left(\hat{u}_n^2 + \Upsilon^2 u_{n,x}^2 + \hat{\theta}_n^2 + \Upsilon^2\theta_{n,x}^2\right) dx} && \text{by applying (36) twice} \\
&= \Upsilon
\end{aligned}$$

which, by arbitrariness of ε, proves the result when $\rho > 0$.

If $\rho = 0$, then $\Upsilon = 1$ and the proof follows the same lines, the only difference being that the Young inequality (37) does not apply for θ and one cannot increase the multiplicative constant of $\theta_{n,x}$ in the numerator. Therefore, the largest lower bound is $\Upsilon = 1$. $\square$

The next lemma provides a crucial estimate for the existence of hylomorphic solitons.

Lemma 20 *Let Λ be as in (8); then*

$$\inf_{\mathbf{u} \in X^+} \Lambda(\mathbf{u}) \leq \min\left\{\sqrt{\frac{1+\sigma}{2}}\,,\ \frac{1+\sigma}{2}\right\}\,.$$

Proof: For any $\varepsilon > 0$ consider a nonnegative and nontrivial function $v \in C^2(\mathbb{R})$ with compact support in $\mathbb{R}$ such that

$$\frac{\int |v''|^2 dx}{\int |v'|^2 dx} < \varepsilon. \tag{38}$$

Such a function exists: if v_0 is any nonnegative and nontrivial function with compact support, then $v(x) = v_0\left(\frac{x}{\mu}\right)$ satisfies (38) for μ sufficiently large.

We first consider

$$\mathbf{u}_R := \left(Rv, 0, Rv', 0\right) \qquad \forall R > 0$$

where v is as in (38). Since $\mathbf{u}_R \in X^+$ for all $R > 0$, we infer the estimate

$$\begin{aligned}
\inf_{\mathbf{u}\in X^+} \Lambda(\mathbf{u}) &\le \frac{E(\mathbf{u}_R)}{C(\mathbf{u}_R)} = \frac{\frac{1}{2}\|\mathbf{u}_R\|^2 + 2\int N(Rv)dx}{C(\mathbf{u}_R)} \\
&= \frac{\frac{1}{2}\int \left[(Rv')^2 + \sigma(Rv')^2 + (Rv'')^2\right] dx + 2\int W(Rv)\,dx}{\int (Rv')^2\, dx} \\
&= \frac{1+\sigma}{2} + \frac{1}{2}\frac{\int |v''|^2 dx}{\int |v'|^2\, dx} + 2\frac{\int W(Rv)dx}{R^2\int |v'|^2\, dx} \\
&< \frac{1+\sigma}{2} + \frac{\varepsilon}{2} + \frac{2}{R^2}\frac{\int W(Rv)\,dx}{\int |v'|^2\, dx} \qquad \text{(by (38))} \\
&\le \frac{1+\sigma}{2} + \frac{\varepsilon}{2} + \frac{2MR^\alpha}{R^2}\frac{\int v^\alpha\, dx}{\int |v'|^2\, dx} \qquad \text{(by (W-iii))}\,.
\end{aligned}$$

By letting $R \to \infty$ and by arbitrariness of ε we finally get

$$\inf_{\mathbf{u}\in X^+} \Lambda(\mathbf{u}) \le \frac{1+\sigma}{2}\,. \tag{39}$$

Then we consider

$$\mathbf{u}_R := \left(Rv, Rv, \sqrt{\tfrac{1+\sigma}{2}}\, Rv', \sqrt{\tfrac{1+\sigma}{2}}\, Rv'\right) \qquad \forall R > 0$$

where v is as in (38): we have that $\mathbf{u}_R \in X^+$ for all $R > 0$. By arguing as above with some crucial changes, we obtain the estimate

$$\begin{aligned}
\inf_{\mathbf{u}\in X^+} \Lambda(\mathbf{u}) &\le \frac{E(\mathbf{u}_R)}{C(\mathbf{u}_R)} = \sqrt{\frac{1+\sigma}{2}} + \frac{1+\rho}{2\sqrt{2(1+\sigma)}}\frac{\int |v''|^2 dx}{\int |v'|^2\, dx} + \frac{\int W(2Rv)dx}{\sqrt{2(1+\sigma)}R^2\int |v'|^2\, dx} \\
&< \sqrt{\frac{1+\sigma}{2}} + \frac{(1+\rho)\varepsilon}{2\sqrt{2(1+\sigma)}} + \frac{M(2R)^\alpha}{\sqrt{2(1+\sigma)}R^2}\frac{\int v^\alpha\, dx}{\int |v'|^2\, dx}\,.
\end{aligned}$$

By letting $R \to \infty$ and by arbitrariness of ε we get

$$\inf_{\mathbf{u}\in X^+} \Lambda(\mathbf{u}) \le \sqrt{\frac{1+\sigma}{2}}\,. \tag{40}$$

The statement follows by combining (39) with (40). □

We now have all the ingredients to prove Theorems 11 and 12.

Proof of Theorem 11. It follows from Theorem 10 with $I = (\beta_0, \infty)$, let us check that all the assumptions are satisfied.

(EC-0) and (EC-1) are trivially satisfied.

(EC-2) follows from Lemma 16.

(EC-3) (i) follows from (W-i)

(EC-3) (ii) and (iii) follow from Lemma 17.

Finally, the hylomorphy condition (11) is satisfied in view of Lemmas 19 and 20 since the condition

$$\min\left\{\sqrt{\frac{1+\sigma}{2}}\,,\ \frac{1+\sigma}{2}\right\} < \min\left\{\sigma + 2\sqrt{2}\,,\ 1 + 2\sqrt{2\rho}\right\}^{1/2}$$

is equivalent to (19).

Therefore, Theorem 10 applies and Theorem 11 is proved. □

Proof of Theorem 12. Since $\mathbf{u}_\beta = \left(u_\beta, \theta_\beta, \hat{u}_\beta, \hat{\theta}_\beta\right) \in X$ is a minimizer, we have $J'_\beta(\mathbf{u}_\beta) = 0$. Then

$$\left(1 + \frac{\beta}{C(\mathbf{u}_\beta)}\right) E'(\mathbf{u}_\beta) - \beta \frac{E(\mathbf{u}_\beta)}{C(\mathbf{u}_\beta)^2} C'(\mathbf{u}_\beta) = 0$$

namely

$$E'(\mathbf{u}_\beta) = \lambda C'(\mathbf{u}_\beta) \tag{41}$$

where

$$\lambda = \lambda_\beta = \frac{\beta \Lambda\left(\mathbf{u}_\beta\right)}{\beta + C(\mathbf{u}_\beta)} > 0\,. \tag{42}$$

If we write (41) explicitly, we get for all $\left(\varphi, \psi, \hat{\varphi}, \hat{\psi}\right) \in X$:

$$\begin{aligned}
\int u''_\beta \varphi'' + \sigma u'_\beta \varphi' + \left[W'(u_\beta + \theta_\beta) + W'(u_\beta - \theta_\beta)\right] \varphi &= \lambda \int \hat{u}_\beta \varphi' \\
\rho \int \theta''_\beta \psi'' + \int \theta'_\beta \psi' + \left[W'(u_\beta + \theta_\beta) - W'(u_\beta - \theta_\beta)\right] \psi &= \lambda \int_\beta \hat{\theta}_\beta \psi' \\
\int \hat{u}_\beta \hat{\varphi} &= \lambda \int u'_\beta \hat{\varphi} \\
\int \hat{\theta}_\beta \hat{\psi} &= \lambda \int \theta'_\beta \hat{\psi}
\end{aligned}$$

namely

$$\begin{aligned}
u''''_\beta - \sigma u''_\beta + W'(u_\beta + \theta_\beta) + W'(u_\beta - \theta_\beta) &= -\lambda \hat{u}'_\beta \\
\rho \theta''''_\beta - \theta''_\beta + W'(u_\beta + \theta_\beta) - W'(u_\beta - \theta_\beta) &= -\lambda \hat{\theta}'_\beta \\
\hat{u}_\beta &= \lambda u'_\beta \\
\hat{\theta}_\beta &= \lambda \theta'_\beta
\end{aligned}$$

so that, after eliminating $\hat{u}_\beta$ and $\hat{\psi}_\beta$, we get

$$\begin{aligned}
u''''_\beta - \sigma u''_\beta + W'(u_\beta + \theta_\beta) + W'(u_\beta - \theta_\beta) &= -\lambda^2 u''_\beta \\
\rho \theta''''_\beta - \theta''_\beta + W'(u_\beta + \theta_\beta) - W'(u_\beta - \theta_\beta) &= -\lambda^2 \theta''_\beta
\end{aligned}$$

that is,

$$\begin{aligned}
u''''_\beta + (\lambda^2 - \sigma) u''_\beta + W'(u_\beta + \theta_\beta) + W'(u_\beta - \theta_\beta) &= 0 \\
\rho \theta''''_\beta + (\lambda^2 - 1) \theta''_\beta + W'(u_\beta + \theta_\beta) - W'(u_\beta - \theta_\beta) &= 0.
\end{aligned}$$

Then we can check directly that the couple

$$u(t, x) = u_\beta(x - \lambda t) \quad , \quad \theta(t, x) = \theta_\beta(x - \lambda t)$$

solves the system (4) with initial conditions

$$\Big(u(0, x), \theta(0, x), u_t(0, x), \theta_t(0, x)\Big) = \Big(u_\beta(x), \theta_\beta(x), -\lambda \hat{u}_\beta, -\lambda \hat{\theta}_\beta\Big)\,.$$

This completes the proof of Theorem 12. □

Note that the condition on σ and ρ in (22) implies (19) so that Theorem 11 guarantees the existence of a hylomorphic soliton $\mathbf{u}_\beta$ for the dynamical system (13) for every $\beta \in I$. However, one more ingredient is needed for the proof of Theorem 13. From [10, 12] we learn that the functional $J_\beta(\mathbf{u})$ restricted to the set $X_0^+ = \{\mathbf{u} \in X^+ \mid \theta = \hat{\theta} = 0\}$ admits a (nontrivial) minimum which we denote by $\mathbf{u}_0 = (u_0, 0, \hat{u}_0, 0)$. We need to prove that $\mathbf{u}_0$ is not a minimum on the whole set X^+. We are able to do so under the assumption (22).

Lemma 21 *Let $\mathbf{u}_0 = (u_0, 0, \hat{u}_0, 0)$ be the minimum of $J_\beta(\mathbf{u})$ restricted to X_0^+. We set*

$$\mathbf{u}_{0\phi} = \left(u_{0\phi}, \theta_{0\phi}, \hat{u}_{0\phi}, \hat{\theta}_{0\phi}\right) := \left(u_0 \cos\phi,\ u_0 \sin\phi,\ \hat{u}_0 \cos\phi,\ \hat{u}_0 \sin\phi\right).$$

If (22) *holds, then*

$$\left(\frac{d^2}{d\phi^2} J_\beta(\mathbf{u}_{0\phi})\right)_{\phi=0} < 0.$$

Proof: In view of (9), for all $\mathbf{u} \in X$ such that $C(\mathbf{u}) > 0$ we have that

$$\begin{aligned} J_\beta(\mathbf{u}) \quad &= \quad \frac{1}{2} A_\beta(\mathbf{u}) \Big[\|\hat{u}\|_{L^2}^2 + \left\|\hat{\theta}\right\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \sigma \|u_x\|_{L^2}^2 + \rho \|\theta_{xx}\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 \\ &\qquad +2 \int \left[W(u+\theta) + W(u-\theta)\right] dx \Big] \end{aligned}$$

where

$$A_\beta(\mathbf{u}) = 1 + \frac{\beta}{\int \left(\hat{u} u_x + \hat{\theta}\theta_x\right) dx}.$$

We observe that, for all ϕ,

$$A_\beta(\mathbf{u}_{0\phi}) = 1 + \frac{\beta}{\int \left[(\hat{u}_0 \cos\phi)(u_{0x}\cos\phi) + (\hat{u}_0 \sin\phi)(u_{0x}\sin\phi)\right] dx} = 1 + \frac{\beta}{\int \hat{u}_0 u_{0x}\, dx} = A_\beta(\mathbf{u}_0) \tag{43}$$

and

$$\|\hat{u}_{0\phi}\|_{L^2}^2 + \left\|\hat{\theta}_{0\phi}\right\|_{L^2}^2 = \cos^2\phi \|\hat{u}_0\|_{L^2}^2 + \sin^2\phi \|\hat{u}_0\|_{L^2}^2 = \|\hat{u}_0\|_{L^2}^2. \tag{44}$$

Then, by (43) and (44), we infer

$$\begin{aligned} J_\beta(\mathbf{u}_{0\phi}) \quad &= \quad \frac{1}{2} A_\beta(\mathbf{u}_0) \Big[\|\hat{u}_0\|_{L^2}^2 + \left(\|u_{0xx}\|_{L^2}^2 + \sigma \|u_{0x}\|_{L^2}^2\right)\cos^2\phi + \left(\rho \|u_{0xx}\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2\right)\sin^2\phi \\ &\qquad +2 \int \left[W(u_0\cos\phi + u_0 \sin\phi) + W(u_0 \cos\phi - u_0\sin\phi)\right] dx \Big]. \end{aligned}$$

Let us compute

$$\begin{aligned} &\frac{d^2}{d\phi^2}\left[\left(\|u_{0xx}\|_{L^2}^2 + \sigma\|u_{0x}\|_{L^2}^2\right)\cos^2\phi + \left((\rho\|u_{0xx}\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2\right)\sin^2\phi\right] \\ &= 2\cos 2\phi\left[-\|u_{0xx}\|_{L^2}^2 - \sigma\|u_{0x}\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2 + (\rho\|u_{0xx}\|_{L^2}^2\right] \\ &= -2\cos 2\phi\left[(1-\rho)\|u_{0xx}\|_{L^2}^2 + (\sigma-1)\|u_{0x}\|_{L^2}^2\right] \end{aligned}$$

so that

$$\begin{aligned} &\frac{d^2}{d\phi^2}\left[\left(\|u_{0xx}\|_{L^2}^2 + \sigma\|u_{0x}\|_{L^2}^2\right)\cos^2\phi + \left(\|u_{0x}\|_{L^2}^2 + (\rho\|u_{0xx}\|_{L^2}^2\right)\sin^2\phi\right]_{\phi=0} \\ &= -2\left[(1-\rho)\|u_{0xx}\|_{L^2}^2 + (\sigma-1)\|u_{0x}\|_{L^2}^2\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} &\frac{d^2}{d\phi^2}\left[W\big((\cos\phi+\sin\phi)u_0\big) + W\big((\cos\phi-\sin\phi)u_0\big)\right] \\ &= \left[W''\big((\cos\phi+\sin\phi)u_0\big)(\cos\phi-\sin\phi)^2 + W''\big((\cos\phi-\sin\phi)u_0\big)(\sin\phi+\cos\phi)^2\right]u_0^2 \\ &+ \left[W'\big((\cos\phi-\sin\phi)u_0\big)(\sin\phi-\cos\phi) - W'\big((\cos\phi+\sin\phi)u_0\big)(\cos\phi+\sin\phi)\right]u_0 \end{aligned}$$

and therefore

$$\frac{d^2}{d\phi^2}\left[W(u_0\cos\phi+u_0\sin\phi)+W(u_0\cos\phi-u_0\sin\phi)\right]_{\phi=0}=2W''(u_0)u_0^2-2W'(u_0)u_0\,.$$

From these two facts we finally obtain that

$$\left(\frac{d^2}{d\phi^2}J_\beta(\mathbf{u}_\phi)\right)_{\phi=0}=$$

$$=-A_\beta(\mathbf{u}_0)\left[(1-\rho)\left\|u_{0xx}\right\|_{L^2}^2+(\sigma-1)\left\|u_{0x}\right\|_{L^2}^2+2\int\left[W'(u_0)u_0-W''(u_0)u_0^2\right]dx\right]<0,$$

where we used (22). □

Proof of Theorem 13. By Theorem 11 it follows that J_β $(\beta\in I)$ has a global minimizer $\mathbf{u}\in X^+$; by Lemma 21, it follows that $\mathbf{u}\notin X_0^+$ and hence it is a torsional soliton. □

Finally, we turn to the proof of Theorem 14 which somehow follows the "dual lines" of the proof of Theorem 13. The condition on σ and ρ in (23) implies (19) so that Theorem 11 guarantees the existence of a hylomorphic soliton $\mathbf{u}_\beta$ for the dynamical system (13) for every $\beta\in I$. From [10, 12] we know that the functional $J_\beta(\mathbf{u})$ restricted to the set $X_1^+=\{\mathbf{u}\in X^+\mid u=\hat{u}=0\}$ admits a nontrivial minimum which we denote by $\Theta_0=(0,\theta_0,0,\hat{\theta}_0)$. We need to prove that Θ_0 is not a minimum on the whole set X^+. We are able to do so under the assumption (23).

Lemma 22 *Let $\Theta_0=(0,\theta_0,0,\hat{\theta}_0)$ be the minimum of $J_\beta(\mathbf{u})$ restricted to X_1^+. We set*

$$\Theta_{0\phi}=\left(u_{0\phi},\theta_{0\phi},\hat{u}_{0\phi},\hat{\theta}_{0\phi}\right):=\left(\theta_0\sin\phi,\ \theta_0\cos\phi,\ \hat{\theta}_0\sin\phi,\ \hat{\theta}_0\cos\phi\right)\,.$$

If (23) *holds, then*

$$\left(\frac{d^2}{d\phi^2}J_\beta(\Theta_{0\phi})\right)_{\phi=0}<0.$$

Proof: As for (43)-(44), we reach the identities

$$A_\beta(\Theta_{0\phi})=A_\beta(\Theta_0)\,,\qquad \left\|\hat{u}_{0\phi}\right\|_{L^2}^2+\left\|\hat{\theta}_{0\phi}\right\|_{L^2}^2=\left\|\hat{\theta}_0\right\|_{L^2}^2\qquad\forall\phi\,. \tag{45}$$

Then, by (45), we infer

$$\begin{aligned}J_\beta(\Theta_{0\phi}) &= \frac{1}{2}A_\beta(\Theta_0)\Bigg[\left\|\hat{\theta}_0\right\|_{L^2}^2+\left(\left\|\theta_{0xx}\right\|_{L^2}^2+\sigma\left\|\theta_{0x}\right\|_{L^2}^2\right)\sin^2\phi+\left(\rho\left\|\theta_{0xx}\right\|_{L^2}^2+\left\|\theta_{0x}\right\|_{L^2}^2\right)\cos^2\phi\Bigg]\\ &\quad +2\int\left[W(\theta_0\cos\phi+\theta_0\sin\phi)+W(\theta_0\sin\phi-\theta_0\cos\phi)\right]dx\Bigg]\,.\end{aligned}$$

Some computations give

$$\begin{aligned}&\frac{d^2}{d\phi^2}\left[\left(\left\|\theta_{0xx}\right\|_{L^2}^2+\sigma\left\|\theta_{0x}\right\|_{L^2}^2\right)\sin^2\phi+\left(\rho\left\|\theta_{0xx}\right\|_{L^2}^2+\left\|\theta_{0x}\right\|_{L^2}^2\right)\cos^2\phi\right]_{\phi=0}\\ &=-2\left[(\rho-1)\left\|\theta_{0xx}\right\|_{L^2}^2+(1-\sigma)\left\|\theta_{0x}\right\|_{L^2}^2\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{d^2}{d\phi^2}\left[W(\theta_0\cos\phi+\theta_0\sin\phi)+W(\theta_0\sin\phi-\theta_0\cos\phi)\right]_{\phi=0}\\ &=W''(\theta_0)\theta_0^2-W'(\theta_0)\theta_0+W''(-\theta_0)(-\theta_0)^2-W'(-\theta_0)(-\theta_0)\,.\end{aligned}$$

From these two facts we finally obtain that

$$\left(\frac{d^2}{d\phi^2}J_\beta(\Theta_\phi)\right)_{\phi=0} = -A_\beta(\Theta_0)\Big[(\rho-1)\left\|\theta_{0xx}\right\|^2_{L^2}+(1-\sigma)\left\|\theta_{0x}\right\|^2_{L^2} + \int\left[W''(\theta_0)\theta_0^2-W'(\theta_0)\theta_0+W''(-\theta_0)(-\theta_0)^2-W'(-\theta_0)(-\theta_0)\right]dx\Big]<0\,,$$

where we used (23). □

Proof of Theorem 14. By Theorem 11 it follows that J_β ($\beta\in I$) has a global minimizer $\mathbf{u}\in X^+$; by Lemma 22, it follows that $\mathbf{u}\notin X_1^+$ and hence it is a longitudinal soliton. □

Acknowledgments. The third author is partially supported by the PRIN project *Equazioni alle derivate parziali di tipo ellittico e parabolico: aspetti geometrici, disuguaglianze collegate, e applicazioni* and by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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