EXISTENCE RESULTS FOR GENERAL CRITICAL GROWTH SEMILINEAR ELLIPTIC EQUATIONS

Filippo Gazzola and Marco Lazzarino

Dipartimento di Scienze e Tecnologie Avanzate,
via Cavour 84, 15100 Alessandria, Italy

ABSTRACT: General critical growth semilinear elliptic equations are considered: existence results are proved according to a suitable "coupled behavior" of the functions involved in the equation.
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1. INTRODUCTION

We prove existence results for the semilinear elliptic problem

$$\begin{cases} -\Delta u = g(x,u) + Q(x)|u|^{2^*-2}u & \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is an open bounded domain with smooth boundary $\partial\Omega$, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, $g(x,\cdot)$ has subcritical growth at infinity (i.e. $\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{2^*-1}} = 0$) and Q is a bounded positive function.

Consider the Hilbert space $H := H_0^1(\Omega)$ endowed with the Dirichlet scalar product; we determine nontrivial solutions of equation (1) as critical points of the functional $J : H \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u) - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*} \ ,$$

where $G(x,s) = \int_0^s g(x,t)dt$. Standard variational arguments do not apply because the embedding $H \subset L^{2^*}(\Omega)$ is not compact, i.e. the functional J does not satisfy the Palais-Smale condition (PS condition). It is well-known that for equations with critical growth, nontrivial solutions may not exist: this follows by suitable applications of Pohožaev identity [10] when Ω is strictly starshaped. In their celebrated paper, Brezis-Nirenberg [2] showed that the functional J satisfies the PS condition in a certain compactness range related to S, the best constant of the embedding $H \subset L^{2^*}(\Omega)$, see [12]: this compactness range is modified according to the maximum value of the function Q, see [4,6]. In a recent paper [8], an orthogonalization technique has been developed for the study of critical growth problems in semilinear elliptic equations; to assure that the considered minimax levels are in the compactness range, certain approximating functions having disjoint support with the Sobolev concentrating functions are constructed. We call Sobolev concentrating functions some truncations of the positive radial entire functions which achieve the best constant in Sobolev inequalities. Let us also mention that the technique developed in [8] has been applied to the study of more general semilinear and quasilinear elliptic problems, see [1,7]. The basic idea of the orthogonalization technique is the following: as we seek critical points of J by a linking argument [11], to prove that the minimax level stays in the compactness range, this requires an estimate of the maximum of the functional J over subsets of $V \oplus \mathbb{R}^+\{u_\varepsilon\}$, where V is some finite dimensional subspace of H and u_ε is the concentrating Sobolev function. Since V and u_ε are not orthogonal, this estimate involves mixed terms which are difficult to estimate even in simple cases, see [3]. An orthogonalization (in H as well as in L^p) of these functions can be obtained by replacing the space V by a space V_ε consisting of functions which approximate the functions in V, but are zero where u_ε is non zero, that is by disjoining their supports. Of course, now the approximation error of V_ε must be estimated, but this can be handled easier than the mentioned mixed terms.

In this paper we show that this orthogonalization technique also applies to more general semilinear elliptic equations as (1); we consider the case where the critical growth term is multiplied by a positive function Q. As we will show, existence results for (1) seem to be related with a "coupled behavior" of the functions Q and g: in simple cases, if such coupled behavior is violated, Pohožaev identity yields nonexistence results, see [5]. Let us also point out that with our assumptions on the lower order term g the functional J may have negative critical levels, and the above mentioned compactness range becomes a nontrivial energy range, see [8]: more precisely, we will prove that a PS sequence within the above energy range yields a nontrivial solution of (1) by means of its weak limit. A preliminary version of the results presented here may be found in [9].

2. STATEMENT OF THE RESULTS

Let λ_k ($k \in \mathbb{N}$), be the eigenvalues of $-\Delta$ relative to the homogeneous Dirichlet problem in Ω; it is well-know that each eigenvalue has finite multiplicity and that $\lambda_k \to +\infty$ as $k \to \infty$.

Without loss of generality we assume that $0 \in \Omega$. We first require the function $g(x,\cdot)$ to be subcritical:

$$\begin{aligned} &g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function satisfying} \\ &\forall \varepsilon > 0 \quad \exists a_\varepsilon \in L^{\frac{2n}{n+2}} \text{ s.t. } |g(x,s)| \le a_\varepsilon(x) + \varepsilon |s|^{\frac{n+2}{n-2}} \text{ for a.e. } x \in \Omega \quad \forall s \in \mathbb{R} \,. \end{aligned} \tag{2}$$

The other assumptions are imposed on the primitive $G(x,s) = \int_0^s g(x,t)dt$: we assume that

$$G(x,s) \ge 0 \quad \text{for a.e. } x \in \Omega \quad \forall s \in \mathbb{R} \tag{3}$$

and that $G(x,\cdot)$ is quadratic in 0. We suppose that Q attains its maximum at 0:

$$\exists M > 0 \text{ s.t. } 0 \le Q(x) \le M \ , \quad Q(0) - Q(x) = M - Q(x) = O(|x|^\beta) \tag{4}$$

where $\beta > 0$ will be chosen differently according to the behavior of $G(x,\cdot)$ at infinity and in 0 (resonance and non-resonance cases).

We assume first that there exist $k \in \mathbb{N}, \delta > 0, \sigma > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

$$\frac{1}{2}(\lambda_k + \sigma)s^2 \leq G(x,s) \leq \frac{1}{2}\mu s^2 \text{ for a.e. } x \in \Omega \quad \forall |s| \leq \delta; \tag{5}$$

furthermore, we assume that

$$G(x,s) \geq \frac{1}{2}(\lambda_k + \sigma)s^2 - \frac{1}{2^*}Q(x)|s|^{2^*} \text{ for a.e. } x \in \Omega \quad \forall s \neq 0. \tag{6}$$

Finally, depending on the values of n and β a growth assumption for G at infinity may be required; we assume that there exists $\Omega_0 \subset \Omega$ with $0 \in \Omega_0$ such that either

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^4} = +\infty \text{ uniformly w.r.t. } x \in \Omega_0 \tag{7}$$

or

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^{k_n^\beta}} = +\infty \quad \text{uniformly w.r.t. } x \in \Omega_0 \text{ for } k_n^\beta = \frac{2(n-\beta)}{n-2}. \tag{8}$$

Depending on the value of β in (4) we make one of the following set of assumptions:

$$\begin{cases} \text{if } n = 3 \text{ and } \beta \geq 1 \text{ assume } (2) - (7) \\ \text{if } n = 3 \text{ and } \beta < 1 \text{ assume } (2) - (6) \text{ and } (8) \\ \text{if } n = 4 \text{ and } \beta \geq 2 \text{ assume } (2) - (6) \\ \text{if } n = 4 \text{ and } \beta < 2 \text{ assume } (2) - (6) \text{ and } (8) \\ \text{if } n \geq 5 \text{ and } \beta > \frac{n}{2} \text{ assume } (2) - (6) \\ \text{if } n \geq 5 \text{ and } \beta \leq \frac{n}{2} \text{ assume } (2) - (6) \text{ and } (8). \end{cases} \tag{9}$$

The existence result in the non-resonance case reads as follows:

Theorem 1 *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a smooth open bounded set and assume (9); then equation (1) admits a nontrivial solution.*

Remark: For all $n \geq 3$ and $\beta > 0$ we have $0 < k_n^\beta < 2^*$ according to (2); moreover, the map $\beta \mapsto k_n^\beta$ is decreasing, that is, more flatness of Q in 0 corresponds to weaker growth assumptions on $G(x,\cdot)$ at infinity. For $n = 3$ we have $k_n^\beta \to 4$ as $\beta \to 1^-$; therefore there is "continuity" between (7) and (8). This continuity is no longer available when $n \geq 4$ because in Lemma 5 below only the behavior of $G(x,\cdot)$ in 0 (and not its behavior at infinity) is used to obtain the crucial estimate (25). For $k_n^\beta = 2$, or equivalently for $\beta = 2$, a nonexistence result of Egnell [5] shows that the growth condition (8) may not be relaxed. □

Let us now assume that there exist $k \in \mathbb{N}$, $\delta > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

$$\frac{1}{2}\lambda_k s^2 \leq G(x,s) \leq \frac{1}{2}\mu s^2 \qquad \text{for a.e. } x \in \Omega \quad \forall |s| \leq \delta; \tag{10}$$

in this case we also assume that there exists $\sigma > 0$ such that

$$G(x,s) \geq \frac{1}{2}\lambda_k s^2 - \Big(\frac{Q(x)}{2^*} - \sigma\Big)|s|^{2^*} \qquad \text{for a.e. } x \in \Omega \quad \forall s \in \mathbb{R} \tag{11}$$

and that there exists an nonempty subset $\Omega_0 \subset \Omega$ with $0 \in \Omega_0$ such that

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^{\frac{8n}{n^2-4}}} = +\infty \text{ uniformly w.r.t. } x \in \Omega_0. \tag{12}$$

We observe that by (2) and (11) there exists $m \in (0, M)$ such that $Q(x) \geq m$ and, necessarily, $\sigma \in (0, \frac{m}{2^*}]$. Finally, we strenghthen (4) with

$$\exists M > 0 \text{ s.t. } 0 \leq Q(x) \leq M\ , \quad Q(0) - Q(x) = M - Q(x) = O(|x|^{n(n-2)/(n+2)})\ . \tag{13}$$

In the resonance case we prove the following

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a smooth open bounded set and assume (2)-(3), (10)-(13); then equation (1) admits a nontrivial solution.*

Remark: In case of resonance, the geometrical properties of the functional become more difficult to derive because the error due to the approximating space is not easily estimated; therefore it seems hard to obtain sharp coupled behavior between G and Q. If $Q(0) - Q(x) = O(|x|^\beta)$ with $\beta < \frac{n(n-2)}{n+2}$, the choice in formula (30) below may not be optimal. □

3. THE VARIATIONAL CHARACTERIZATION

We recall that a sequence $\{u_m\} \subset H$ is called a PS sequence for a functional I at level c if $I(u_m) \to c$ and $I'(u_m) \to 0$ in H^{-1}. The functional I satisfies the PS condition at level c, if every PS sequence at level c is precompact in H. As already mentioned, the functional J does not satisfy the PS condition; however, the following result holds:

Lemma 1 *Under the assumptions of Theorem 1 or of Theorem 2, let $\{u_m\} \subset H$ be a PS sequence for J; then, there exists $u \in H$ such that $u_m \rightharpoonup u$ up to a subsequence and $J'(u) = 0$. Moreover, if $J(u_m) \to c$ with $c \in \Big(0, \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}\Big)$ then $u \not\equiv 0$ and hence u is a nontrivial solution of (1).*

Proof. The proof is standard: we only briefly sketch it. Let $f(x,s) = g(x,s) + Q(x)|s|^{2^*-2}s$ and $F(x,s) = \int_0^s f(x,t)dt$; since (2) holds, we have

$$\exists \mu > 2 \quad \exists \bar{s} \geq 0 \qquad \text{such that} \qquad \mu F(x,s) \leq f(x,s)s \qquad \text{for a.e. } x \in \Omega \quad \forall |s| \geq \bar{s}\ .$$

Therefore $\{u_m\}$ is bounded and there exists u such that $u_m \rightharpoonup u$, up to a subsequence. Furthermore, $J'(u) = 0$ by weak continuity of J'. Assume $c \in \Big(0, \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}\Big)$ and, by contradiction, $u \equiv 0$. As the term $g(x,u_m)u_m$ is subcritical, by (4) we infer

$$\begin{aligned} o(1) = J'(u_m)[u_m] &= \int_\Omega |\nabla u_m|^2 - \int_\Omega g(x,u_m)u_m - \int_\Omega Q(x)|u_m|^{2^*} \\ &\geq \|u_m\|^2 - M\|u_m\|_{2^*}^{2^*} + o(1)\ . \end{aligned} \tag{14}$$

By definition of S we have $\|u\|^2 \geq S\|u\|_{2^*}^2$ for all $u \in H$: therefore,

$$o(1) \geq \|u_m\|^2(1 - S^{-\frac{2^*}{2}} M \|u_m\|^{2^*-2}).$$

If $\|u_m\| \to 0$ we contradict $c > 0$, therefore $\|u_m\|^2 \geq \frac{S^{\frac{n}{2}}}{M^{\frac{n-2}{2}}} + o(1)$ and by (2),(4) and (14) we get

$$J(u_m) \geq \frac{1}{n}\|u_m\|^2 + \frac{n-2}{2n}(\|u_m\|^2 - M\|u_m\|_{2^*}^{2^*}) + o(1) \geq \frac{1}{n}\frac{S^{\frac{n}{2}}}{M^{\frac{n-2}{2}}} + o(1)$$

which contradicts $c < \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}$. □

Next, consider the family of entire functions

$$u_\varepsilon^*(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}} \tag{15}$$

and define a positive cut-off function $\eta \in C_c^\infty(B_{1/m})$ (where B_r denotes the ball of radius r with center in 0) such that $\eta \equiv 1$ in $B_{1/2m}$, $\eta \leq 1$ in $B_{1/m}$ and $\|\nabla\eta\|_\infty \leq 4m$; consider the sequence of functions $u_\varepsilon(x) := \eta(x) \cdot u_\varepsilon^*(x)$. As $\varepsilon \to 0$ we have the following estimates (see [2]):

$$\|u_\varepsilon\|^2 = S^{\frac{n}{2}} + O(\varepsilon^{n-2}) \qquad \|u_\varepsilon\|_{2^*}^{2^*} = S^{\frac{n}{2}} + O(\varepsilon^n). \tag{16}$$

Furthermore, we can prove

Lemma 2 *If m is large enough and $Q(x)$ satisfies (4) then, for $\varepsilon \to 0$*

$$\int_\Omega Q(x)|u_\varepsilon|^{2^*} = MS^{\frac{n}{2}} + O(\bar{h}(\varepsilon)) \qquad \text{with } \bar{h}(\varepsilon) = \begin{cases} \varepsilon^\beta & \text{for } \beta < n \\ \varepsilon^n|\ln\varepsilon| & \text{for } \beta = n \\ \varepsilon^n & \text{for } \beta > n. \end{cases} \tag{17}$$

Proof. By (4) we can choose m large enough so that $M - Q(x) \leq c|x|^\beta$ for a.e. $x \in B_{1/m}$. Then, by the hypotheses on η, (4) and (16) we have

$$\begin{aligned} \int_\Omega Q(x)|u_\varepsilon|^{2^*} &= \int_{B_{1/m}} M|u_\varepsilon|^{2^*} + \int_{B_{1/m}} [Q(x) - M]|u_\varepsilon|^{2^*} \\ &\geq MS^{\frac{n}{2}} + O(\varepsilon^n) - c\int_{B_{1/m}} |x|^\beta |u_\varepsilon|^{2^*}. \end{aligned}$$

Next, take $\varepsilon < \frac{1}{m}$ and evaluate

$$\int_{B_{1/m}} |x|^\beta |u_\varepsilon|^{2^*} \leq \int_{B_{1/m}} |x|^\beta |u_\varepsilon^*|^{2^*} \ ;$$

to this end we estimate separately

$$\int_{B_\varepsilon} \frac{|x|^\beta \varepsilon^n}{(\varepsilon^2 + |x|^2)^n} \leq \frac{1}{\varepsilon^n}\int_{B_\varepsilon} |x|^\beta = C_1\varepsilon^\beta$$

and

$$\int_{B_{1/m}\setminus B_\varepsilon} \frac{|x|^\beta \varepsilon^n}{(\varepsilon^2+|x|^2)^n} \leq c\varepsilon^n \int_\varepsilon^{1/m} \rho^{\beta-2n}\rho^{n-1} \leq \begin{cases} C_2\varepsilon^\beta & \text{for } \beta<n \\ C_2\varepsilon^n|\ln\varepsilon| & \text{for } \beta=n \\ C_2\varepsilon^n & \text{for } \beta>n\ : \end{cases}$$

then (17) follows. □

Let e_i be an L^2 normalized eigenvector relative to the eigenvalue λ_i, let

$$H^- := \text{span}\{e_i; i=1,...,k\}\ ,$$

$H^+ := (H^-)^\perp$ and let $P_k : H \to H^-$ denote the orthogonal projection.

We take m large enough so that $B_{2/m} \subset \Omega_0$. Consider the functions $\zeta_m : \Omega \to \mathbb{R}$ defined by

$$\zeta_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m} \\ m|x|-1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } x \in \Omega \setminus B_{2/m}\ , \end{cases} \tag{18}$$

the "approximating eigenfunctions" $e_i^m := \zeta_m e_i$ and the space

$$H_m^- := \text{span}\{e_i^m; i=1,...,k\}.$$

For all $v \in H_m^- \oplus \mathbb{R}^+\{u_\varepsilon\}$ we can write $v = w + \alpha u_\varepsilon$, where, by definition,

$$\text{supp}(u_\varepsilon) \cap \text{supp}(w) = \emptyset\ . \tag{19}$$

As $m \to \infty$ we have (see Lemma 2 in [8])

$$e_i^m \to e_i \text{ in } H \quad \textit{and} \quad \max_{u\in H_m^-; \int u^2=1} \|u\|^2 \leq \lambda_k + c_k m^{2-n}\ . \tag{20}$$

Let us prove that the functional J has a linking geometrical structure (see [11]).

Proposition 1 *Under the assumptions of Theorems 1 or 2 there exist $\alpha, \rho > 0$ such that*

$$J(v) \geq \alpha \qquad \forall v \in \partial B_\rho \cap H^+\ . \tag{21}$$

Moreover, there exists $R > \rho$ such that if $Q_m^\varepsilon = [(B_R \cap H_m^-) \oplus [0,R]\{u_\varepsilon\}]$ then

$$\max_{v\in\partial Q_m^\varepsilon} J(v) \leq \omega_m \tag{22}$$

with $\omega_m \to 0$ as $m \to \infty$. Finally, if m is large enough, $\partial B_\rho \cap H^+$ and ∂Q_m^ε link.

Proof. By (2) and either (5) or (10) we infer that there exists $C > 0$ such that

$$G(x,s) \leq \frac{1}{2}\mu s^2 + C|s|^{2^*}$$

and therefore

$$J(v) \geq C_1\|v\|^2 - C_2\|v\|^{2^*} \qquad \forall v \in H^+$$

with $C_1, C_2 > 0$ and (21) follows.

By (20) and either (6) or (11) we clearly have

$$\lim_{m\to\infty} \max_{v\in H_m^-} J(v) = 0 \ .$$

By (3) we have $J(ru_\varepsilon) \leq \frac{r^2}{2}\|u_\varepsilon\|^2 - \frac{r^{2^*}}{2^*}\int_\Omega Q(x)|u_\varepsilon|^{2^*}$ which, by (16) becomes negative if $r = R$ and R is large enough. Therefore,

$$J(v) \leq \omega_m \qquad \forall v \in (H_m^-) \cup (H_m^- \oplus R\{u_\varepsilon\}) \tag{23}$$

with $\omega_m \to 0$ as $m \to \infty$. Since $\max_{0\leq r\leq R} J(ru_\varepsilon) < +\infty$, by (19) and for R large enough, we obtain

$$J(v) \leq 0 \qquad \forall v \in [(\partial B_R \cap H_m^-) \oplus [0,R]\{u_\varepsilon\}] \ ; \tag{24}$$

(22) follows by (23) and (24).

We complete the proof observing that by (20), if m is large enough, then

$$P_k H_m^- = H^- \quad \text{ and } \quad H_m^- \oplus H^+ = H$$

where $P_k : H \to H^-$ is the projection introduced above: therefore, $\partial B_\rho \cap H^+$ and ∂Q_m^ε link. □

By Proposition 1, the functional J satisfies all the assumptions of the linking Theorem except for the PS condition. The proofs of Theorem 1 and 2 are performed by constructing a PS sequence for J at level $c \in \left(0, \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}\right)$; indeed, in such case, Lemma 1 yields a nontrivial solution of (1).

We will proceed as follows: let $\Gamma = \{h \in C(\bar{Q}_m^\varepsilon, H); h(v) = v, \ \forall v \in \partial Q_m^\varepsilon\}$, then by standard methods we obtain a PS sequence for J at level

$$c = \inf_{h\in\Gamma} \max_{v\in Q_m^\varepsilon} J(h(v)) \ .$$

By (21) we obviously have $c > 0$; moreover, since the identity $Id \in \Gamma$, we have $c \leq \max_{v\in Q_m^\varepsilon} J(v)$: Theorems 1 and 2 follow if we prove that for ε small enough

$$\max_{v\in Q_m^\varepsilon} J(v) < \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} \ . \tag{25}$$

4. PROOF OF THEOREM 1

Throughout this section we assume that (9) holds. Choose m large enough so that

$$c_k m^{2-n} < \sigma \tag{26}$$

where c_k is as in (20) and σ is as in (5). As the set $\{v \in Q_m^\varepsilon; J(v) \geq 0\}$ is compact, for all $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that for $v_\varepsilon = w_\varepsilon + t_\varepsilon u_\varepsilon$, we have

$$J(v_\varepsilon) = \max_{v\in Q_m^\varepsilon} J(v) = \frac{1}{2}\|v_\varepsilon\|^2 - \int_\Omega G(x, v_\varepsilon) - \frac{1}{2^*}\int_\Omega Q(x)|v_\varepsilon|^{2^*}.$$

By Proposition 1, we immediately obtain that the sequences $\{t_\varepsilon\} \subset \mathbb{R}^+$ and $\{w_\varepsilon\} \subset H_m^-$ are bounded: hence, up to subsequences, we may assume that

$$t_\varepsilon \to t_0 \geq 0 \qquad \text{and} \qquad w_\varepsilon \to w_0 \in H_m^-$$

where the convergence of $\{w_\varepsilon\}$ can be viewed in any norm since the space H_m^- is finite dimensional. Note that by (6), (20) and (26) we have

$$J(w_\varepsilon) \leq 0 \qquad \forall w_\varepsilon \in H_m^- \ . \tag{27}$$

Let us now prove

Lemma 3 *As $\varepsilon \to 0$ we have*

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)|u_\varepsilon|^{2^*} \leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} + O(\varepsilon^\alpha) \ \textit{with}\ \alpha = \min(\beta, n-2) \ .$$

Proof. The derivative of the function $f(x) = ax^2 - bx^{2^*}$ vanishes for $x_{max} = [\frac{a(n-2)}{bn}]^{\frac{n-2}{4}}$ and $f(x_{max}) = \frac{2a}{n}[\frac{a(n-2)}{bn}]^{\frac{n-2}{2}}$; then, by taking

$$a = \frac{1}{2}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) \qquad b = \frac{1}{2^*}MS^{\frac{n}{2}} + O(\bar{h}(\varepsilon))$$

(with $\bar{h}(\varepsilon)$ as in (17)) we obtain

$$f(x_{max}) \leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} + O(\varepsilon^\alpha) \qquad \text{with } \alpha = \min(\beta, n-2).$$

Hence by (16) and Lemma 2 we have

$$\frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)|u_\varepsilon|^{2^*} \leq \frac{1}{2}(S^{\frac{n}{2}} + O(\varepsilon^{n-2}))t_\varepsilon^2 - \frac{1}{2^*}(MS^{\frac{n}{2}} + O(\bar{h}(\varepsilon)))t_\varepsilon^{2^*}$$

$$\leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} + O(\varepsilon^\alpha).$$

□

If $t_\varepsilon \to 0$ we have $J(v_\varepsilon) \to c \leq 0$ and we are done: hence, from now on, suppose that $t_\varepsilon \to t_0 > 0$. To estimate the lower order term $\int_\Omega G(x, t_\varepsilon u_\varepsilon)$ several cases must be considered.

Lemma 4 *If (8) holds, then there exists a function $\tau_1 = \tau_1(\varepsilon)$ such that $\lim_{\varepsilon\to 0} \tau_1(\varepsilon) = +\infty$ and such that, for ε small enough we have*

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \tau_1(\varepsilon)\varepsilon^\beta \ .$$

Proof. For ε small enough we have $B_\varepsilon \subset B_{1/2m} \subset \Omega_0$ and by (3) and (15) we infer

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \int_{B_\varepsilon} G\Big(x, t_\varepsilon \frac{[c\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}\Big) \ .$$

By (8) there exists $\bar{s} > 0$ and an increasing function $\phi = \phi(s)$ with $\lim_{s\to+\infty} \phi(s) = +\infty$ such that if $s \geq \bar{s}$ then

$$G(x,s) \geq \phi(s)s^{k_n^\beta} \qquad \text{for a.e. } x \in \Omega_0. \tag{28}$$

Next, note that if ε is small enough (recall that $t_\varepsilon \to t_0 > 0$) then

$$t_\varepsilon \frac{[c\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}} > \bar{s} \quad \forall x \in B_\varepsilon;$$

hence, by (28) we get

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq c \int_{B_\varepsilon} \phi(c\varepsilon^{\frac{2-n}{2}})(\varepsilon^{\frac{2-n}{2}})^{k_n^\beta} \geq c\phi(c\varepsilon^{\frac{2-n}{2}})\varepsilon^{\frac{2-n}{2}k_n^\beta + n} = c\phi(c\varepsilon^{\frac{2-n}{2}})\varepsilon^\beta$$

and the assertion follows by taking $\tau_1(\varepsilon) = c\phi(\varepsilon^{\frac{2-n}{2}})$. □

For the cases in (9) where (8) does not hold we obtain:

Lemma 5 *There exists a function $\tau_2 = \tau_2(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0} \tau_2(\varepsilon) = +\infty$ and, for ε small enough,*

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \tau_2(\varepsilon)\varepsilon^\alpha\ , \qquad \alpha = \min(\beta, n-2)\ .$$

Proof. If $n = 3$, by (7) one can reason like in Lemma 4 with $k_n^\beta = 4$ and obtain $\tau_2(\varepsilon) = c\phi(\varepsilon^{-\frac{1}{2}})$ for a suitable ϕ.

If $n \geq 4$ one can use the behavior of $G(x,\cdot)$ at 0 and reason like in Lemma 5 in [8]: then one obtains
- if $n = 4$ take $\tau_2(\varepsilon) = c|\ln\varepsilon|$
- if $n \geq 5$ and $\beta \geq n-2$ (i.e. $\alpha = n-2$) take $\tau_2(\varepsilon) = \varepsilon^{2-n/2}$
- if $n \geq 5$ and $\beta \in \left(\frac{n}{2}, n-2\right)$ (i.e. $\alpha = \beta$) take $\tau_2(\varepsilon) = \varepsilon^{n/2-\beta}$. □

The proof of Theorem 1 is now easily completed by (19), (27) and Lemmas 3-5; indeed we have

$$J(v_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon) \leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} + (c - \tau_i(\varepsilon))\varepsilon^\alpha < \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}$$

for ε small enough with $i = 1, 2$ according to the different cases.

5. PROOF OF THEOREM 2

In this section we assume hypotheses (2)-(4), (10)-(13). In order to emphasize the dependence on m we denote $u_\varepsilon^m, w_\varepsilon^m, v_\varepsilon^m$ instead $u_\varepsilon, w_\varepsilon, v_\varepsilon$ (this dependence is hidden in the cut-off function η). As in the previous section we want to show (25): for all $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that for $v_\varepsilon = w_\varepsilon + tu_\varepsilon$ we have

$$J(v_\varepsilon) = \max_{v\in Q_m^\varepsilon} J(v) = \frac{1}{2}\|v_\varepsilon\|^2 - \int_\Omega G(x, v_\varepsilon) - \frac{1}{2^*}\int_\Omega Q(x)|v_\varepsilon|^{2^*}.$$

We first remark that we can again reduce to the case where the sequences $\{t_\varepsilon\}$ and $\{w_\varepsilon^m\}$ satisfy

$$t_\varepsilon \geq c > 0, \qquad \|w_\varepsilon^m\| \leq c. \tag{29}$$

Taking into account the dependence on m, we obtain

Lemma 6 *Let $m \to \infty$ and assume that $\varepsilon = \varepsilon(m) = o(\frac{1}{m})$; then*

$$\|u_\varepsilon^m\|^2 = S^{\frac{n}{2}} + O[(\varepsilon m)^{n-2}] \qquad \int_\Omega Q(x)|u_\varepsilon^m|^{2^*} = MS^{\frac{n}{2}} + O[(\varepsilon m)^n + \varepsilon^{n(n-2)/(n+2)}] \ .$$

Moreover, there exists a function Φ such that $\lim_{x\to\infty} \Phi(x) = +\infty$ and

$$\int_\Omega G(x, u_\varepsilon^m) \geq \varepsilon^{\frac{n(n-2)}{n+2}} \Phi(\varepsilon^{-1}) \ .$$

Proof. The estimates of $\|u_\varepsilon^m\|^2$ and $\int_\Omega G(x, u_\varepsilon^m)$ are just Lemma 6 in [8]. By reasoning as in Lemma 2 with $\beta = \frac{n(n-2)}{n+2}$ (and hence $\bar{h}(\varepsilon) = \varepsilon^{n(n-2)/(n+2)}$) we get

$$\int_\Omega Q(x)|u_\varepsilon^m|^{2^*} = MS^{\frac{n}{2}} + O[(\varepsilon m)^n + \varepsilon^{n(n-2)/(n+2)}]$$

and the lemma is proved. □

Now we take $\varepsilon = \varepsilon(m)$ with

$$\varepsilon(m) = m^{-\frac{(n+2)}{2}} \ ; \tag{30}$$

hence, for $m \to \infty$, $\varepsilon(m) = o(\frac{1}{m})$ and Lemma 6 applies. From now on, we denote by v^m, u^m, w^m the functions $v_\varepsilon^m, w_\varepsilon^m, u_\varepsilon^m$ with the above choice of ε and with t_m the corresponding t_ε. Let Φ be the function defined in Lemma 6, then by reasoning as for Lemmas 7 and 8 in [8], for m large enough, we obtain

$$J(t_m u^m) \leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} - cm^{\frac{n(2-n)}{2}} \Phi(cm^{\frac{n+2}{2}}) \tag{31}$$

and

$$J(w^m) \leq cm^{\frac{n(2-n)}{2}} \ . \tag{32}$$

The proof of Theorem 2 is now obtained by (19), (31) and (32):

$$J(v^m) = J(t_m u^m) + J(w^m) \leq \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}} - cm^{\frac{n(2-n)}{2}} (\Phi(m^{\frac{n+2}{2}}) - 1) < \frac{S^{\frac{n}{2}}}{nM^{\frac{n-2}{2}}}$$

for m large enough.

REFERENCES

[1] G. Arioli and F. Gazzola, Some results on p-Laplace equations with a critical growth term, to appear on *Diff. Int. Eq.*

[2] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.,* **36** (1983), 437-477

[3] A. Capozzi, D. Fortunato and G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, *Ann. Inst. H. Poincaré A.N.L.,* **2, 6** (1985), 463-470

[4] A. Capozzi and G. Palmieri, Multiplicity results for nonlinear elliptic equations involving critical Sobolev exponent, *Proc. Roy. Soc. Edinburgh,* **103A** (1986), 275-285

[5] H. Egnell, Semilinear elliptic equations involving critical Sobolev exponents, *Arch. Rat. Mech. Anal.,* **104** (1988), 27-56

[6] J.F. Escobar, Positive solutions for some semilinear elliptic equations with critical Sobolev exponents, *Comm. Pure Appl. Math.,* **40** (1987), 623-657

[7] F. Gazzola, Critical growth problems for polyharmonic operators, to appear on *Proc. Roy. Soc. Edinburgh*

[8] F. Gazzola and B. Ruf, Lower order perturbations of critical growth nonlinearities in semilinear elliptic equations, *Adv. Diff. Eq.,* **4** (1997), 555-572

[9] M. Lazzarino, *Su alcuni problemi ellittici semilineari con generico termine a crescita critica,* Tesi di Laurea, Alessandria, 1997

[10] S.J. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Doklady,* **6** (1965), 1408-1411

[11] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Reg. Conf. Series Math. 65, Amer. Math. Soc., Providence, R.I., 1986

[12] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.,* **110** (1976), 353-372