ON THE FIRST EIGENVALUE OF A FOURTH ORDER STEKLOV PROBLEM

DORIN BUCUR, ALBERTO FERRERO, FILIPPO GAZZOLA

Abstract. We prove some results about the first Steklov eigenvalue d_1 of the biharmonic operator in bounded domains. Firstly, we show that Fichera's principle of duality [9] may be extended to a wide class of nonsmooth domains. Next, we study the optimization of d_1 for varying domains: we disprove a long-standing conjecture, we show some new and unexpected features and we suggest some challenging problems. Finally, we prove several properties of the ball.

1. Introduction

For any open bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary, consider the fourth order Steklov boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u - d u_\nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $d \in \mathbb{R}$ and u_ν denotes the outer normal derivative of u on $\partial\Omega$. By a solution of (1) we mean a function $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = d \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \forall v \in H^2 \cap H^1_0(\Omega). \tag{2}$$

An eigenvalue of (1) is a value of d for which (2) admits nontrivial solutions, the corresponding eigenfunctions. Let $d_1(\Omega)$ be defined by

$$d_1(\Omega) = \inf_u \frac{\displaystyle\int_\Omega |\Delta u|^2 \, dx}{\displaystyle\int_{\partial\Omega} u_\nu^2 dS} \tag{3}$$

where the infimum is taken over all functions $u \in \left[H^2 \cap H^1_0(\Omega)\right] \setminus H^2_0(\Omega)$. If the infimum in (3) is achieved then $d_1(\Omega)$ is the first (smallest) eigenvalue of (1) and the corresponding minimizer u is the first eigenfunction.

Elliptic problems with parameters in the boundary conditions are called *Steklov problems* from their first appearance in [25]. In the case of the biharmonic operator, these conditions were first considered by Kuttler-Sigillito [17] and Payne [21] who studied the isoperimetric properties of the first eigenvalue d_1. As pointed out by Kuttler [15, 16], d_1 is the sharp constant for L^2 a priori

1991 *Mathematics Subject Classification.* 35J40, 65L15.

Key words and phrases. biharmonic operator, first Steklov eigenvalue, shape optimization.

D.B. Laboratoire de Mathématiques CNRS UMR 5127, Université de Savoie, 73376 Le-Bourget-Du-Lac, France (dorin.bucur@univ-savoie.fr)

A.F. Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, 20125 Milano, Italy (alberto.ferrero@unimib.it)

F.G. Dipartimento di Matematica, Politecnico di Milano, 20133 Milano, Italy (filippo.gazzola@polimi.it).

estimates for solutions of the (second order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. More recently, the whole spectrum of the biharmonic Steklov problem was studied in [8] where one can also find a physical interpretation of d_1 and of the Steklov boundary conditions. We also refer to [4, 5, 10] for some related nonlinear problems and for the study of the positivity preserving property of the biharmonic operator under Steklov boundary conditions. In this paper we study the first Steklov eigenvalue d_1 from several points of view.

In Section 2.1 we state that a function $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ which achieves equality in (3) exists provided the domain Ω is either smooth (C^2) or satisfies a geometric condition which is fulfilled if Ω has no "reentrant corners" (for instance, if Ω is convex).

With a suitable scaling, one sees that $d_1(k\Omega) = k^{-1}d_1(\Omega)$ for any bounded domain Ω and any $k > 0$ so that $d_1(k\Omega) \to 0$ as $k \to \infty$. This fact suggests that $d_1(\Omega)$ becomes "smaller" when the domain Ω becomes "larger". Problem 1.9 in [8] raises the question whether the map $\Omega \mapsto d_1(\Omega)$ is monotone decreasing with respect to domain inclusion. On one hand, in view of the validity of such property for several "similar" maps (for instance, the first Dirichlet eigenvalue of $-\Delta$), it would be reasonable to expect a positive answer. On the other hand, since functions in the space $H^2 \cap H^1_0(\Omega)$ allow no truncations and no trivial extensions outside Ω, it also appears reasonable to expect a negative answer. In Section 2.2 we show that the answer is negative.

Due to the above mentioned homogeneity, one is then led to seek domains which minimize $d_1(\Omega)$ under suitable constraints, the most natural one being the *volume constraint*. It is known since Faber-Krahn [7, 13, 14] that under such constraint the minimizer for the first Dirichlet eigenvalue of $-\Delta$ is a ball. Smith [23] stated that the same holds true for d_1, at least for planar domains. But, as noticed by Kuttler and Sigillito, the argument in [23] contains a gap, see the "Note added in proof" at p.111 in [24]. A few years later, Kuttler [15] proved that a (planar) square has a first Steklov eigenvalue d_1 which is strictly smaller than the one of the disk having the same measure; the estimate by Kuttler was recently improved in [8]. Therefore, it is *not true* that $d_1(\Omega^*) \leq d_1(\Omega)$ where Ω^* denotes the spherical rearrangement of Ω. For this reason, Kuttler [15] suggested a different minimization problem with a *perimeter constraint*; in [15, Formula (11)] he conjectures that a planar disk minimizes d_1 among all domains having fixed perimeter. He brings numerical evidence that on rectangles his conjecture seems true, see also [18]. In Section 2.2 we show that also this conjecture is false and that no optimal shape for d_1 exists since its infimum is zero under perimeter constraint in any space dimension $n \geq 2$. Our argument shows that cylinders with "small holes" have arbitrarily small d_1. In Problem 1 we suggest a new different optimization problem under the *convexity constraint*.

The question of stability of the first eigenvalue for small geometric perturbations of the disk is discussed in Section 2.3. In Theorem 6 we prove that the first eigenvalue of the Steklov problem on circumscribed regular polygons converges to the first eigenvalue of the disk, when the number of edges goes to infinity, hence no "Babuska paradox" holds. Finally, we state that, although the ball has no isoperimetric property, it is a stationary domain for the map $\Omega \mapsto d_1(\Omega)$ in the class of C^4 domains under smooth perturbations which preserve measure.

This paper is organized as follows. In the next section we state our main results; those are divided in three subsections (existence of minimizers, shape optimization, stability and stationarity of the ball). In Section 3 we set up the functional analytic framework. Sections 4-9 are devoted to the proofs of the main results.

2. Main results

2.1. **Existence of minimizers.** We start with a definition taken from [1]:

Definition 1. *We say that an open domain* $\Omega \subset \mathbb{R}^n$ *satisfies the outer ball condition if for each* $p \in \partial\Omega$ *there exists an open ball* $B \subset \mathbb{R}^n \setminus \Omega$ *such that* $p \in \partial B$*. We say that it satisfies the uniform outer ball condition if the radius of the ball* B *can be taken independently of* $p \in \partial\Omega$*.*

It is clear that if $\partial\Omega$ is smooth (C^2) or if Ω is convex, then Ω satisfies the uniform outer ball condition. The following existence result for a minimizer of $d_1(\Omega)$ holds true:

Theorem 1. *Assume that* $\Omega \subset \mathbb{R}^n$ *is an open bounded domain with Lipschitz boundary which satisfies the uniform outer ball condition. Then* $d_1(\Omega)$ *admits a positive minimizer* $u \in [H^2 \cap H^1_0(\Omega)] \backslash H^2_0(\Omega)$ *which is unique up to a constant multiplier.*

Theorem 1 is already known in the case $\partial\Omega \in C^2$, see [4, 8, 9]. As far as we are aware, there are no counterexamples to the existence of a minimizer for $d_1(\Omega)$ when Ω does not satisfy the uniform outer ball condition. However, in view of [19], we believe that a minimizer might not exist in domains with a *concave corner*.

Next, we recall from [8, 9] an alternative characterization of $d_1(\Omega)$. Let

$$C^2_H\left(\overline{\Omega}\right) := \left\{v \in C^2\left(\overline{\Omega}\right);\ \Delta v = 0 \text{ in } \Omega\right\}$$

and consider the norm defined by $\|v\|_H := \|v\|_{L^2(\partial\Omega)}$ for all $v \in C^2_H\left(\overline{\Omega}\right)$. Then define

$$\mathbf{H} := \text{the completion of } C^2_H\left(\overline{\Omega}\right) \text{ with respect to the norm } \|\cdot\|_H .$$

Since Ω is assumed to have a Lipschitz boundary, by [12] we infer that $\mathbf{H} \subset H^{1/2}(\Omega) \subset L^2(\Omega)$. Therefore, the quantity

$$\delta_1(\Omega) := \inf_{h \in \mathbf{H}\backslash\{0\}} \frac{\displaystyle\int_{\partial\Omega} h^2 dS}{\displaystyle\int_{\Omega} h^2 dx}$$

is well defined. This minimization problem was previously studied in [8, 9] assuming that $\partial\Omega \in C^2$. Here we prove

Theorem 2. *If* $\Omega \subset \mathbb{R}^n$ *is an open bounded domain with Lipschitz boundary, then* $\delta_1(\Omega)$ *admits a minimizer* $h \in \mathbf{H}\backslash\{0\}$*. If we also assume that* Ω *satisfies the uniform outer ball condition then this minimizer is positive, unique up to a constant multiplier and* $\delta_1(\Omega) = d_1(\Omega)$*.*

2.2. **Shape optimization.** We are here interested in studying the map $\Omega \mapsto d_1(\Omega)$ when Ω varies in suitable classes of domains. We first consider a class of cylinders:

Theorem 3. *Let* $D_\varepsilon = \{x \in \mathbb{R}^2; \varepsilon < |x| < 1\}$ *and let* $\Omega_\varepsilon \subset \mathbb{R}^n$ *(*$n \geq 2$*) be such that*

$$\Omega_\varepsilon = D_\varepsilon \times (0,1)^{n-2} ;$$

in particular, if $n = 2$ *we have* $\Omega_\varepsilon = D_\varepsilon$*. Then,*

$$\lim_{\varepsilon \to 0^+} d_1(\Omega_\varepsilon) = 0.$$

This statement has several important consequences. Firstly, it shows that $d_1(\Omega)$ has no optimal shape under the constraint that Ω is contained in a fixed ball:

Corollary 1. *Let* $B_R = \{x \in \mathbb{R}^n; |x| < R\}$*. Then, for any* $R > 0$

$$\inf_{\Omega \subseteq B_R} d_1(\Omega) = 0$$

where the infimum is taken over all domains $\Omega \subseteq B_R$ *such that* $\partial\Omega \in C^\infty$ *if* $n = 2$ *and* $\partial\Omega$ *is Lipschitzian if* $n \geq 3$*.*

The difference of regularity between dimensions $n = 2$ and $n \geq 3$ is that $\partial\Omega_\varepsilon \in C^\infty$ whenever $n = 2$ while $\partial\Omega_\varepsilon$ is just Lipschitzian whenever $n \geq 3$; in the latter case, Ω_ε satisfies the uniform outer ball condition with radius $R = \varepsilon$.

A second consequence of Theorem 3 is that it disproves a conjecture by Kuttler [15] which states that the disk has the smallest d_1 among all planar regions having the same perimeter; this forces us to propose two alternative problems suggested by Theorem 3 and Corollary 1:

Problem 1. *Denote by B the unit ball in $\mathbb{R}^n$. Consider the following minimization problems:*

$$\inf_{\Omega\in\mathcal{M}_B} d_1(\Omega) \tag{4}$$

where $\mathcal{M}_B$ is the family of all convex domains $\Omega \subset \mathbb{R}^n$ such that $|\Omega| = |B|$ and

$$\inf_{\Omega\in\mathcal{P}_B} d_1(\Omega) \tag{5}$$

where $\mathcal{P}_B$ is the family of all convex domains $\Omega \subset \mathbb{R}^n$ such that $|\partial\Omega| = |\partial B|$.

Does there exist an optimal shape for the minimization problems (4) *and* (5)*? If an optimal shape for* (4) *exists, we know it is not the ball.*

Theorem 3 also gives an answer to Problem 1.9 in [8] and shows that the map $\Omega \mapsto d_1(\Omega)$ is *not* monotone decreasing with respect to domain inclusion.

Finally, Theorem 3 raises several natural questions. Why do we consider an annulus in the plane and the region between two cylinders in space dimensions $n \geq 3$? What happens if we consider an annulus in any space dimension? The quite surprising answer is given in

Theorem 4. *Let $n \geq 3$ and let $\Omega^\varepsilon = \{x \in \mathbb{R}^n;\ \varepsilon < |x| < 1\}$.*
(i) If $n = 3$ then

$$\lim_{\varepsilon\to0^+} d_1(\Omega^\varepsilon) = 2.$$

(ii) If $n \geq 4$ then

$$\lim_{\varepsilon\to0^+} d_1(\Omega^\varepsilon) = n.$$

Theorems 3 and 4 highlight a striking difference between dimension $n = 2$, dimension $n = 3$ and dimensions $n \geq 4$. This difference may find some explanation in the *capacity* of a domain whose behaviour strictly depends on the space dimension. But more surprises are in order... Since the set Ω^ε is smooth, by Theorem 2 it follows that $d_1(\Omega^\varepsilon) = \delta_1(\Omega^\varepsilon)$. Moreover, since our proof of Theorem 4 uses radial harmonic functions $h = h(r)$ $(r = |x|)$, we may rewrite the ratio defining $\delta_1(\Omega^\varepsilon)$ as

$$\frac{\displaystyle\int_{\partial\Omega^\varepsilon} h^2 dS}{\displaystyle\int_{\Omega^\varepsilon} h^2 dx} = \frac{(h(1))^2 + \varepsilon^{n-1}(h(\varepsilon))^2}{\displaystyle\int_\varepsilon^1 (h(r))^2 r^{n-1} dr}\ .$$

In this setting, we can treat the space dimension n as a real number. Then, we prove

Theorem 5. *Let $\varepsilon \in (0,1)$, let $K_\varepsilon = \{h \in C^2([\varepsilon,1]);\ h''(r) + \frac{n-1}{r}h'(r) = 0,\ r \in [\varepsilon,1]\}$ and, for all $n \in [1,\infty)$, let*

$$\gamma_\varepsilon(n) = \inf_{h\in K_\varepsilon\setminus\{0\}} \frac{(h(1))^2 + \varepsilon^{n-1}(h(\varepsilon))^2}{\displaystyle\int_\varepsilon^1 (h(r))^2 r^{n-1} dr}\ .$$

Then,

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon(n)=\begin{cases}2 & \text{if } n=1\\ 0 & \text{if } 1<n<3\\ 2 & \text{if } n=3\\ n & \text{if } n>3\ .\end{cases}$$

Theorem 5 shows that dimensions $n = 1$ and $n = 3$ are "discontinuous" dimensions for the behaviour of γ_ε. This is due to the asymptotic behaviour of some trial functions, see the proof. But we have no physical explanation of this fact.

2.3. **Stability and stationarity of the ball.** The convergence of the spectrum of elliptic operators with Dirichlet boundary conditions on varying domains can be handled, in general, via the Mosco convergence of the corresponding functional spaces, see [6, Chapters 4-6]. In our case two difficulties occur: on the one hand the spaces under consideration are $H^2\cap H^1_0(P_k)$ and, in view of Babuska's paradox [3], it is not clear whether a suitable Mosco convergence holds for the entire spaces and, on the second hand, the Steklov boundary condition (producing a boundary integral in the denominator of the Rayleigh quotient) requires a *strong* geometric convergence (namely a very fine topology) in order to preserve the perimeter.

We show that we do have stability of the first eigenvalue on the sequence of regular polygons converging to the disk:

Theorem 6. *Let $n = 2$ and let $\{P_k\}$ be a sequence of regular polygons with k edges circumscribed to the unit disk D centered at the origin. Then*

$$\lim_{k\to\infty} d_1(P_k) = d_1(D) = 2.$$

We stated Theorem 6 in a particular situation because in this case the computations can be performed explicitly and because it is of some interest to compare this result with Babuska paradox. Nevertheless, we expect that the continuity of d_1 is ensured on more general sequences of convex bounded domains converging uniformly (in the Hausdorff topology) to a bounded convex domain.

For any multi-index $\alpha = (\alpha_1, ...\alpha_n) \in \mathbb{N}^n$ let $|\alpha| = \sum_i \alpha_i$ and for any real smooth function u defined in $\mathbb{R}^n$, let

$$\partial^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\ \partial x_n^{\alpha_n}}.$$

Then, for any $k \geq 1$ denote by

$$C^k_b(\mathbb{R}^n;\mathbb{R}^n)=\left\{\theta=(\theta_1,...,\theta_n):\partial^\alpha\theta_i\in C^0\cap L^\infty(\mathbb{R}^n)\ \text{ for any } 1\leq i\leq n \text{ and } 0\leq|\alpha|\leq k\right\}$$

the Banach space endowed with the norm

$$\|\theta\|_{C^k_b}=\max_{\substack{0\leq|\alpha|\leq k\\ 1\leq i\leq n}}\ \sup_{x\in\mathbb{R}^n}|\partial^\alpha\theta_i(x)|\,.$$

The next statement shows continuity of the map $\Omega\mapsto d_1(\Omega)$ under smooth perturbations:

Theorem 7. *The map $\Omega\mapsto d_1(\Omega)$ is continuous with respect to C^2 diffeomorphism of $\mathbb{R}^n$ in the sense that for any fixed domain Ω_0 with C^2 boundary we have: for any $\varepsilon>0$ there exists $\delta>0$ such that for any $\theta\in C^2_b(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\theta\|_{C^2_b}<\delta$ we have $|d_1((I+\theta)(\Omega_0))-d_1(\Omega_0)|<\varepsilon$. Here I denotes the identity map in $\mathbb{R}^n$.*

Finally, let us explain what we mean by a *stationary* domain:

Definition 2. *Let $k \geq 1$ and let $\Omega_0 \subset \mathbb{R}^n$ be an open bounded domain with $\partial\Omega_0 \in C^k$. We say that Ω_0 is a stationary domain with respect to $C_b^k(\mathbb{R}^n;\mathbb{R}^n)$ volume preserving deformations if for any map $\gamma \in C^1\left([0,1];C_b^k(\mathbb{R}^n;\mathbb{R}^n)\right)$ such that*

$$\gamma(0) = I \quad \text{and} \qquad |\gamma(t)(\Omega_0)| = |\Omega_0| \qquad \forall t \in [0,1],$$

we have

$$\frac{d}{dt} d_1\left(\gamma(t)(\Omega_0)\right)_{|t=0} = 0.$$

We prove the following

Theorem 8. *The unit ball $B \subset \mathbb{R}^n$ is a stationary domain with respect to $C_b^4(\mathbb{R}^n;\mathbb{R}^n)$ volume preserving deformations.*

The volume preserving assumption in Theorem 8 is crucial: indeed, we know that $kd_1(kB) = d_1(B)$ so that the unit ball is not a stationary domain with respect to such a kind of deformations.

3. Preliminaries

We first endow the space $H^2 \cap H_0^1(\Omega)$ with a Hilbert structure:

Lemma 1. *Assume that Ω is a Lipschitz bounded domain which satisfies the uniform outer ball condition. Then the space $H^2 \cap H_0^1(\Omega)$ becomes a Hilbert space when endowed with the scalar product*

$$(u,v) := \int_\Omega \Delta u \Delta v \, dx \qquad \forall u, v \in H^2 \cap H_0^1(\Omega). \tag{6}$$

Proof. Since $H^2 \cap H_0^1(\Omega)$ is a closed subspace of $H^2(\Omega)$, it is a Hilbert space when endowed with the scalar product of $H^2(\Omega)$. In view of the assumptions made on Ω, we know that elliptic regularity estimates hold for the second order Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

In particular, if $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is a solution to (7), then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2} \leq C \|f\|_{L^2}$$

for a suitable constant C independent of f, see [1]. Hence, by the Closed Graph Theorem it follows that the norm defined by (6) is equivalent to the norm induced by $H^2(\Omega)$ so that $H^2 \cap H_0^1(\Omega)$ is a Hilbert space also when endowed with (6). □

We now consider the following linear variational problem: given $F \in \left(H^2 \cap H_0^1(\Omega)\right)'$ find $u \in H^2 \cap H_0^1(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx - d \int_{\partial\Omega} u_\nu v_\nu \, dS = \langle F, v \rangle \qquad \forall v \in H^2 \cap H_0^1(\Omega). \tag{8}$$

Since Ω is a bounded Lipschitz domain, by [20, Theorem 6.2, Chapter 2], we deduce that the boundary integral in (8) makes sense and that the linear map

$$\begin{array}{rcl} H^2 \cap H_0^1(\Omega) & \to & \left(L^2(\partial\Omega)\right)^n \\ u & \mapsto & \nabla u|_{\partial\Omega} \end{array} \tag{9}$$

is well defined and compact. On the other hand, the normal derivative to a Lipschitz domain is defined almost everywhere on $\partial\Omega$ so that $u_\nu \in L^2(\partial\Omega)$ for any $u \in H^2 \cap H_0^1(\Omega)$. Then, we can prove

Lemma 2. *Let $F \in \left(H^2 \cap H_0^1(\Omega)\right)'$. Then problem (8) admits a solution $u \in H^2 \cap H_0^1(\Omega)$ if and only if $\langle F, u_* \rangle = 0$ for any solution u_* of (2).*

Proof. Let T, Z be the linear operators implicitly defined by

$$\langle Tu, v\rangle = \int_\Omega \Delta u \Delta v \; dx \qquad \forall u, v \in H^2 \cap H_0^1(\Omega)$$

$$\langle Zu, v\rangle = \int_{\partial\Omega} u_\nu v_\nu \; dS \qquad \forall u, v \in H^2 \cap H_0^1(\Omega).$$

Then, $T \in \mathcal{L}\left(H^2 \cap H_0^1(\Omega); (H^2 \cap H_0^1(\Omega))'\right)$, namely T is a linear continuous operator. By the Riesz Representation Theorem in Hilbert spaces we know that T is an isomorphism, i.e. T^{-1} exists and $T^{-1} \in \mathcal{L}\left((H^2 \cap H_0^1(\Omega))'; H^2 \cap H_0^1(\Omega)\right)$. Moreover, the compactness of the map (9) implies that Z is a compact linear operator from $H^2 \cap H_0^1(\Omega)$ into $\left(H^2 \cap H_0^1(\Omega)\right)'$.

Next, consider the compact linear self-adjoint operator $K : H^2 \cap H_0^1(\Omega) \to H^2 \cap H_0^1(\Omega)$ defined by $K = T^{-1}Z$. If we denote by I the identity in $H^2 \cap H_0^1(\Omega)$ we have for any $d \neq 0$ and $\mu = d^{-1}$

$$K - \mu I = -\mu T^{-1} (T - dZ) . \tag{10}$$

Problem (8) is equivalent to find $u \in H^2 \cap H_0^1(\Omega)$ such that $(T - dZ)\, u = F$ and, by (10), to $(K - \mu I)\, u = -\mu T^{-1}F$. By the Fredholm alternative applied to K we infer that this equation is solvable if and only if $T^{-1}F \in$Ker$(K^* - \mu I)^\perp =$Ker$(K - \mu I)^\perp$. In view of (10), this means

$$\left(T^{-1}F, u_*\right) = 0 \qquad \forall u_* \in \text{Ker}\,(K - \mu I) = \text{Ker}\,(T - dZ)$$

and, in turn,

$$\langle F, u_*\rangle = 0 \qquad \forall u_* \in \text{Ker}\,(T - dZ) .$$

Finally, it is clear that $u_* \in$Ker$(T - dZ)$ if and only if u_* solves (2). This completes the proof. □

4. Proof of Theorems 1-2

Proof of Theorem 1. Let $\{u_m\}$ be a minimizing sequence for $d_1(\Omega)$ with $\|\Delta u_m\|_{L^2(\Omega)} = 1$ so that $\{u_m\}$ is bounded in $H^2(\Omega)$. Up to a subsequence, we may assume that there exists $u \in H^2 \cap H_0^1(\Omega)$ such that $u_m \rightharpoonup u$ in $H^2(\Omega)$, see Lemma 1. Then, since Ω is Lipschitzian and satisfies the uniform outer ball condition, the map in (9) is compact and we deduce that $(u_m)_\nu \to u_\nu$ in $L^2(\partial\Omega)$.

On the other hand, since $\{u_m\}$ is a minimizing sequence, $\|\Delta u_m\|_{L^2(\Omega)} = 1$ and $\|(u_m)_\nu\|_{L^2(\partial\Omega)}$ is bounded then $d_1(\Omega) > 0$, u_ν is not identically zero on $\partial\Omega$ and

$$\|u_\nu\|_{L^2(\partial\Omega)}^{-2} = \lim_{m\to\infty} \|(u_m)_\nu\|_{L^2(\partial\Omega)}^{-2} = d_1(\Omega) .$$

Moreover, by weak lower semicontinuity of the norm, we also have

$$\|\Delta u\|_{L^2(\Omega)}^2 \le \liminf_{m\to\infty} \|\Delta u_m\|_{L^2(\Omega)}^2 = 1$$

and hence $u \in \left[H^2 \cap H_0^1(\Omega)\right] \setminus H_0^2(\Omega)$ satisfies

$$\frac{\|\Delta u\|_{L^2(\Omega)}^2}{\|u_\nu\|_{L^2(\partial\Omega)}^2} \le d_1(\Omega) .$$

This proves that u is a minimizer for $d_1(\Omega)$. Uniqueness up to a constant multiplier follows by arguing as in [4]. □

Proof of Theorem 2. In the first part of this proof, we just assume that Ω is a domain with Lipschitz boundary. Let $\{h_m\} \subset \mathbf{H}\setminus\{0\}$ be a minimizing sequence for $\delta_1(\Omega)$ with $\|h_m\|_H = \|h_m\|_{L^2(\partial\Omega)} = 1$. Up to a subsequence, we may assume that there exists $h \in \mathbf{H}$ such that $h_m \rightharpoonup h$ in $\mathbf{H}$. By regularity estimates [11, 12], we infer that there exists a constant $C > 0$ such that

$$\|h\|_{H^{1/2}(\Omega)} \le C\,\|h\|_{L^2(\partial\Omega)} \qquad \forall h \in \mathbf{H}$$

so that the sequence $\{h_m\}$ is bounded in $H^{1/2}(\Omega)$, $h_m \rightharpoonup h$ in $H^{1/2}(\Omega)$ up to a subsequence and, by compact embedding, we also have $h_m \to h$ in $L^2(\Omega)$. Therefore, since $\{h_m\}$ is a minimizing sequence, $\|h_m\|_{L^2(\partial\Omega)} = 1$ and $\|h_m\|_{L^2(\Omega)}$ is bounded then $\delta_1(\Omega) > 0$, $h \in \mathbf{H}\setminus\{0\}$ and

$$\|h\|^{-2}_{L^2(\Omega)} = \lim_{m\to\infty} \|h_m\|^{-2}_{L^2(\Omega)} = \delta_1(\Omega).$$

Moreover, by weak lower semicontinuity of $\|\cdot\|_H$ we also have

$$\|h\|^2_{L^2(\partial\Omega)} = \|h\|^2_H \le \liminf_{m\to\infty} \|h_m\|^2_H = 1$$

and hence $h \in \mathbf{H}\setminus\{0\}$ satisfies

$$\frac{\|h\|^2_{L^2(\partial\Omega)}}{\|h\|^2_{L^2(\Omega)}} \le \delta_1(\Omega).$$

This proves that h is a minimizer for $\delta_1(\Omega)$.

In the rest of the proof, we make the further assumption that Ω satisfies the outer ball condition. Under this condition, by Theorem 1 we have the existence of a minimizer for $d_1(\Omega)$. The fact that $\delta_1(\Omega) = d_1(\Omega)$ follows by arguing as in [8, Section 5]: in particular, there is a one-to-one correspondence between minimizers of $\delta_1(\Omega)$ and $d_1(\Omega)$ so that uniqueness of a minimizer for $\delta_1(\Omega)$ up to a constant multiplier follows from Theorem 1. □

5. Proof of Theorem 3

For any $\varepsilon \in (0,1)$ let $w_\varepsilon \in H^2 \cap H^1_0(D_\varepsilon)$ be defined by

$$w_\varepsilon(x) = \frac{1-|x|^2}{4} - \frac{1-\varepsilon^2}{4\log\varepsilon}\log|x| \qquad \forall x \in D_\varepsilon. \tag{11}$$

Then we have

$$\Delta w_\varepsilon = -1 \qquad \text{in } \Omega_\varepsilon$$

and

$$|\nabla w_\varepsilon(x)|^2 = \left(\frac{|x|}{2} + \frac{1-\varepsilon^2}{4\log\varepsilon}\frac{1}{|x|}\right)^2 \qquad \forall x \in \overline{\Omega}_\varepsilon$$

so that

$$\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2\, dx = \pi\left(1-\varepsilon^2\right)$$

and

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} (w_\varepsilon)^2_\nu\; dS &= 2\pi\left(\frac{1}{2} + \frac{1-\varepsilon^2}{4\log\varepsilon}\right)^2 + 2\pi\varepsilon\left(\frac{\varepsilon}{2} + \frac{1-\varepsilon^2}{4\varepsilon\log\varepsilon}\right)^2 \\ &= \frac{\pi}{8}\frac{1}{\varepsilon\log^2\varepsilon} + o\left(\frac{1}{\varepsilon\log^2\varepsilon}\right) \to +\infty \qquad \text{as } \varepsilon \to 0^+.\end{aligned} \tag{12}$$

It follows immediately that

$$\lim_{\varepsilon\to 0^+} d_1(\Omega_\varepsilon) \le \lim_{\varepsilon\to 0^+} \frac{\displaystyle\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2\, dx}{\displaystyle\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2\ dS} = 0.$$

This completes the proof of the theorem for $n=2$.

We now consider the case $n \ge 3$. Let

$$u_\varepsilon(x) = \left(\prod_{i=3}^n x_i(1-x_i)\right) w_\varepsilon(x_1,x_2) \qquad \forall x\in\Omega_\varepsilon$$

where w_ε is as in (11); note that u_ε vanishes on $\partial\Omega_\varepsilon$ and $u_\varepsilon \in H^2\cap H^1_0(\Omega_\varepsilon)$. Then, we have

$$\Delta u_\varepsilon = -\prod_{i=3}^n x_i(1-x_i) - 2w_\varepsilon(x_1,x_2)\sum_{j=3}^n \prod_{\substack{i=3\\ i\neq j}}^n x_i(1-x_i)$$

(with the convention that $\prod_{i\in\emptyset}\beta_i = 1$) and

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2\, dx \le 2\int_{\Omega_\varepsilon}\prod_{i=3}^n x_i^2(1-x_i)^2\, dx + 8\int_{\Omega_\varepsilon} w_\varepsilon^2(x_1,x_2)\sum_{j=3}^n\prod_{\substack{i=3\\ i\neq j}}^n x_i^2(1-x_i)^2\ dx\ ;$$

hence, since $|w_\varepsilon(x)| < \frac{1}{2}$ for all $x\in D_\varepsilon$, there exists $C>0$ such that

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2\, dx \le C \qquad \forall \varepsilon\in(0,1)\ . \tag{13}$$

On the other hand, we have

$$|\nabla u_\varepsilon|^2 = \prod_{i=3}^n x_i^2(1-x_i)^2\left[\left(\frac{\partial w_\varepsilon}{\partial x_1}\right)^2 + \left(\frac{\partial w_\varepsilon}{\partial x_2}\right)^2\right] + \sum_{j=3}^n\left((1-2x_j)^2 w_\varepsilon^2(x_1,x_2)\prod_{\substack{i=3\\ i\neq j}}^n x_i^2(1-x_i)^2\right)$$

and since w_ε vanishes on ∂D_ε we obtain

$$\int_{\partial\Omega_\varepsilon}(u_\varepsilon)_\nu^2\, dS = \int_{\partial\Omega_\varepsilon}|\nabla u_\varepsilon|^2\, dS \ge \int_{\partial D_\varepsilon\times(0,1)^{n-2}}|\nabla u_\varepsilon|^2\, dS$$

$$\ge \int_{\partial D_\varepsilon}(w_\varepsilon)_\nu^2\, dS\cdot\prod_{i=3}^n\int_0^1 x_i^2(1-x_i)^2\, dx_i \to +\infty$$

as $\varepsilon\to 0^+$ in view of (12). Therefore, by (13) we obtain

$$\lim_{\varepsilon\to 0^+} d_1(\Omega_\varepsilon) \le \lim_{\varepsilon\to 0^+}\frac{\displaystyle\int_{\Omega_\varepsilon}|\Delta u_\varepsilon|^2\, dx}{\displaystyle\int_{\partial\Omega_\varepsilon}(u_\varepsilon)_\nu^2\, dS} = 0$$

which proves the theorem also when $n\ge 3$.

6. Proof of Theorems 4-5

In Theorem 4 we assume that $n \geq 3$ is an integer, so that by Theorem 2 we know that $d_1(\Omega^\varepsilon) = \delta_1(\Omega^\varepsilon)$ for any $\varepsilon \in (0,1)$ and that $\delta_1(\Omega^\varepsilon)$ admits a unique minimizer h_ε up to a constant multiplier. By the symmetric structure of Ω^ε, we deduce that h_ε is necessarily radially symmetric. Moreover by Theorem 1.8 in [8], h_ε is not a constant function. Therefore

$$\delta_1(\Omega^\varepsilon) = \inf_h \frac{\displaystyle\int_{\partial\Omega^\varepsilon} h^2 dS}{\displaystyle\int_{\Omega^\varepsilon} h^2 dx}$$

where the infimum is taken among all radial functions $h \in C^2\left(\overline{\Omega^\varepsilon}\right)$ which are harmonic in Ω^ε. If we put $r = |x|$ then any radial harmonic function $h = h(r)$ belongs to the space K_ε and $\delta_1(\Omega^\varepsilon) = \gamma_\varepsilon(n)$ for integer n, where K_ε and γ_ε are defined in the statement of Theorem 5. Therefore, if we prove Theorem 5, also Theorem 4 follows.

Assume that $n \geq 1$ and $n \neq 2$, the case $n = 2$ being already established in Theorem 3. It is straightforward that any nonconstant $h \in K_\varepsilon$, up to a constant multiplier, has the form

$$h_a(r) = r^{2-n} + a \qquad \forall r \in [\varepsilon, 1], \tag{14}$$

for some $a \in \mathbb{R}$. Hence, if we define

$$N_\varepsilon(a) := (h_a(1))^2 + \varepsilon^{n-1}(h_a(\varepsilon))^2, \qquad D_\varepsilon(a) := \int_\varepsilon^1 (h_a(r))^2 r^{n-1} dr ,$$

then, by direct computation we obtain

$$N_\varepsilon(a) = \left(1 + \varepsilon^{n-1}\right) a^2 + 2(1+\varepsilon)a + (1 + \varepsilon^{3-n}), \tag{15}$$

$$D_\varepsilon(a) = \int_\varepsilon^1 (r^{3-n} + 2ar + a^2 r^{n-1})\, dr = \begin{cases} \frac{1-\varepsilon^n}{n} a^2 + (1-\varepsilon^2)a + \frac{1-\varepsilon^{4-n}}{4-n} & \text{if } n \neq 4 \\ \frac{1-\varepsilon^4}{4} a^2 + (1-\varepsilon^2)a - \log\varepsilon & \text{if } n = 4 \, . \end{cases} \tag{16}$$

Let $g_\varepsilon(a) := \frac{N_\varepsilon(a)}{D_\varepsilon(a)}$ so that

$$\gamma_\varepsilon(n) = \min_{a\in\mathbb{R}} g_\varepsilon(a) \, .$$

To study this minimization problem, we need the following simple fact: let $\alpha, \beta, \gamma, \lambda, \mu, \nu \in \mathbb{R}$, then

$$\varphi(s) = \frac{\alpha s^2 + \beta s + \gamma}{\lambda s^2 + \mu s + \nu} \quad \Longrightarrow \quad \varphi'(s) = \frac{(\alpha\mu - \beta\lambda)s^2 + 2(\alpha\nu - \gamma\lambda)s + (\beta\nu - \gamma\mu)}{(\lambda s^2 + \mu s + \nu)^2} \, . \tag{17}$$

In the rest of this proof we distinguish several cases according to the value of n.

The cases $3 < n < 4$ and $n > 4$. According to (15)-(16), in this case we have

$$g_\varepsilon(a) = \frac{\left(1 + \varepsilon^{n-1}\right) a^2 + 2(1+\varepsilon)a + (1 + \varepsilon^{3-n})}{\frac{1-\varepsilon^n}{n} a^2 + (1-\varepsilon^2)a + \frac{1-\varepsilon^{4-n}}{4-n}}$$

and, by (17), we have

$$g'_\varepsilon(a) = 0 \iff A_\varepsilon a^2 + B_\varepsilon a + C_\varepsilon = 0 \, , \tag{18}$$

where

$$A_\varepsilon := (1 + \varepsilon^{n-1})(1 - \varepsilon^2) - \frac{2(1+\varepsilon)(1-\varepsilon^n)}{n} \, , \tag{19}$$

$$B_\varepsilon := \frac{2(1+\varepsilon^{n-1})(1-\varepsilon^{4-n})}{4-n} - \frac{2(1+\varepsilon^{3-n})(1-\varepsilon^{n})}{n} \ , \tag{20}$$

$$C_\varepsilon := \frac{2(1+\varepsilon)(1-\varepsilon^{4-n})}{4-n} - (1+\varepsilon^{3-n})(1-\varepsilon^{2}) \ . \tag{21}$$

Since $A_\varepsilon > 0$ for $\varepsilon < 1$, (18) shows that g_ε achieves its global minimum at

$$a_\varepsilon := \frac{-B_\varepsilon + \sqrt{B_\varepsilon^2 - 4A_\varepsilon C_\varepsilon}}{2A_\varepsilon}.$$

Then, as $\varepsilon \to 0^+$, we have

$$A_\varepsilon = \frac{n-2}{n} + o(1) \ , \qquad B_\varepsilon = -\frac{2}{n}\varepsilon^{3-n} + o\left(\varepsilon^{3-n}\right) \ , \qquad C_\varepsilon = -\varepsilon^{3-n} + o\left(\varepsilon^{3-n}\right) \ .$$

Since $A_\varepsilon > 0$ for $\varepsilon < 1$, (18) shows that g_ε achieves its global minimum at

$$a_\varepsilon = \frac{-B_\varepsilon + \sqrt{B_\varepsilon^2 - 4A_\varepsilon C_\varepsilon}}{2A_\varepsilon} = \frac{2}{n-2}\varepsilon^{3-n} + o\left(\varepsilon^{3-n}\right).$$

Finally, we obtain

$$g_\varepsilon\left(a_\varepsilon\right) = \frac{(1+\varepsilon^{n-1})a_\varepsilon^2 + 2(1+\varepsilon)a_\varepsilon + (1+\varepsilon^{3-n})}{\frac{1-\varepsilon^n}{n}a_\varepsilon^2 + (1-\varepsilon^2)a_\varepsilon + \frac{1-\varepsilon^{4-n}}{4-n}} = \frac{\frac{4}{(n-2)^2}\varepsilon^{6-2n} + o(\varepsilon^{6-2n})}{\frac{4}{n(n-2)^2}\varepsilon^{6-2n} + o\left(\varepsilon^{6-2n}\right)}$$

so that

$$\lim_{\varepsilon\to 0^+}\gamma_\varepsilon(n) = \lim_{\varepsilon\to 0^+} g_\varepsilon(a_\varepsilon) = n \qquad \forall n \in (3,4)\cup(4,\infty).$$

The case $n = 4$. In this case, by (15) and (16), we obtain

$$g_\varepsilon(a) = \frac{(1+\varepsilon^3)a^2 + 2(1+\varepsilon)a + (1+\varepsilon^{-1})}{\frac{1-\varepsilon^4}{4}a^2 + (1-\varepsilon^2)a - \log\varepsilon}$$

and, according to (17), we have again (18) but now with

$$A_\varepsilon := (1+\varepsilon^3)(1-\varepsilon^2) - \frac{(1+\varepsilon)(1-\varepsilon^4)}{2} = \frac{1}{2} + o(1) \ ,$$

$$B_\varepsilon := -2(1+\varepsilon^3)\log\varepsilon - \frac{(1+\varepsilon^{-1})(1-\varepsilon^4)}{2} = -\frac{1}{2}\varepsilon^{-1} + o\left(\varepsilon^{-1}\right) \ ,$$

$$C_\varepsilon := -2(1+\varepsilon)\log\varepsilon - (1+\varepsilon^{-1})(1-\varepsilon^2) = -\varepsilon^{-1} + o\left(\varepsilon^{-1}\right) \ ,$$

as $\varepsilon \to 0^+$. Since $A_\varepsilon > 0$ for $\varepsilon < 1$, we know that g_ε attains its minimum at

$$a_\varepsilon = \frac{-B_\varepsilon + \sqrt{B_\varepsilon^2 - 4A_\varepsilon C_\varepsilon}}{2A_\varepsilon} = \frac{\frac{1}{2}\varepsilon^{-1} + o(\varepsilon^{-1}) + \sqrt{\frac{1}{4}\varepsilon^{-2} + o(\varepsilon^{-2})}}{1 + o(1)} = \varepsilon^{-1} + o(\varepsilon^{-1}).$$

Hence,

$$g_\varepsilon(a_\varepsilon) = \frac{(1+\varepsilon^3)a_\varepsilon^2 + 2(1+\varepsilon)a_\varepsilon + (1+\varepsilon^{-1})}{\frac{1-\varepsilon^4}{4}a_\varepsilon^2 + (1-\varepsilon^2)a_\varepsilon - \log\varepsilon} = \frac{(1+\varepsilon^3)(\varepsilon^{-2} + o(\varepsilon^{-2}))}{\frac{1-\varepsilon^4}{4}\varepsilon^{-2} + o(\varepsilon^{-2})}$$

and

$$\lim_{\varepsilon\to 0^+}\gamma_\varepsilon(4) = \lim_{\varepsilon\to 0^+} g_\varepsilon(a_\varepsilon) = 4.$$

The case $n = 3$**.** In order to compute $N_\varepsilon(a)$ and $D_\varepsilon(a)$, it is sufficient to replace $n = 3$ into (15) and (16). Also A_ε, B_ε, C_ε may be obtained by replacing $n = 3$ into (19) (20) (21). But now the asymptotic estimates as $\varepsilon \to 0^+$ become

$$A_\varepsilon = \frac{1}{3} + o(1) \ , \qquad B_\varepsilon = \frac{2}{3} + o(1) \ ,$$

while $C_\varepsilon = 0$ for any $\varepsilon \in (0, 1)$. As in the previous cases, we infer that g_ε achieves its global minimum at $a_\varepsilon = 0$ and

$$\lim_{\varepsilon\to 0^+} \gamma_\varepsilon(3) = \lim_{\varepsilon\to 0^+} g_\varepsilon(0) = \lim_{\varepsilon\to 0^+} \frac{2}{1-\varepsilon} = 2.$$

The cases $1 < n < 2$ **and** $2 < n < 3$**.** In these cases, we obtain the following asymptotic expansions as $\varepsilon \to 0^+$:

$$A_\varepsilon = \frac{n-2}{n} + o(1) \ , \qquad B_\varepsilon = \frac{4(n-2)}{n(4-n)} + o(1) \ , \qquad C_\varepsilon = \frac{n-2}{4-n} + o(1) \ .$$

Note that $A_\varepsilon, B_\varepsilon, C_\varepsilon > 0$ if $2 < n < 3$ whereas $A_\varepsilon, B_\varepsilon, C_\varepsilon < 0$ if $1 < n < 2$. However, in both these situations we have

$$a_\varepsilon = \frac{-B_\varepsilon + \sqrt{B_\varepsilon^2 - 4A_\varepsilon C_\varepsilon}}{2A_\varepsilon} = -1 + o(1)$$

as $\varepsilon \to 0^+$. Therefore, we obtain

$$\lim_{\varepsilon\to 0^+} \gamma_\varepsilon(n) = \lim_{\varepsilon\to 0^+} g_\varepsilon(a_\varepsilon) = 0 \qquad \forall n \in (1, 2) \cup (2, 3).$$

The case $n = 1$**.** In this case, we have $A_\varepsilon \equiv 0$ and, as $\varepsilon \to 0^+$:

$$B_\varepsilon = -\frac{2}{3} + o(1) \ , \qquad C_\varepsilon = -\frac{1}{3} + o(1) \ .$$

Therefore, the function g_ε admits a maximum for $a = -C_\varepsilon/B_\varepsilon = -1/2 + o(1)$ and no minimum. This fact has a simple explanation: the function h_a introduced in (14) is not correct if $n = 1$ since minimizers for δ_1 in intervals are constants, see [4, 8] for the details. This corresponds to the case $a = \infty$ in (14). Since $g_\varepsilon(a)$ tends to 2 at infinity, we have

$$\lim_{\varepsilon\to 0^+} \gamma_\varepsilon(1) = 2.$$

7. Proof of Theorem 6

We start with the following

Lemma 3. *Let $\{P_k\}$ be the sequence of polygons as in the statement of Theorem* 6*. If $u \in H^1_0(P_k)$ and $\Delta u \in L^2(P_k)$ then $u \in H^2(P_k)$ and moreover there exists a constant $C > 0$ independent of u and k such that*

$$\|u\|_{H^2(P_k)} \le C\|\Delta u\|_{L^2(P_k)}.$$

Proof. Using the notations of [1], for any $k \ge 3$ we define the function β_k such that

$$P_k = \{(x, y) \in \mathbb{R}^2;\ (x, y) = (r\cos\theta, r\sin\theta)\ ,\ 0 \le r < \beta_k(\theta)\ ,\ 0 \le \theta < 2\pi\}$$

and the radii of the inscribed and circumscribed disk:

$$\rho_{0k} = \min_{\theta\in[0,2\pi)} \beta_k(\theta), \qquad \rho_{1k} = \max_{\theta\in[0,2\pi)} \beta_k(\theta).$$

We prove some uniform estimates of the Lipschitz constant for β_k. It is not restrictive assuming that one of the vertices of the polygon P_k lies on the x_1 axis so that it is enough to study the function β_k in the interval $[0, 2\pi/k]$. We have

$$\beta_k(\theta) = \frac{\sqrt{1+\tan^2\theta}}{\cos\frac{\pi}{k}\left(\tan\frac{\pi}{k}\tan\theta + 1\right)} \qquad \forall\theta \in \left[0, \frac{2\pi}{k}\right].$$

By elementary computations one sees that

$$\beta_k'(\theta) = \frac{\tan\theta - \tan\frac{\pi}{k}}{\cos\frac{\pi}{k}\cos^2\theta \cdot \sqrt{1+\tan^2\theta}\cdot\left(\tan\frac{\pi}{k}\tan\theta+1\right)^2} \qquad \forall\theta \in \left(0, \frac{2\pi}{k}\right)$$

and

$$\beta_k''(\theta) = \frac{\left(\tan^2\frac{\pi}{k}+2\right)\tan^2\theta - 2\tan\frac{\pi}{k}\tan\theta + 2\tan^2\frac{\pi}{k}+1}{\cos\frac{\pi}{k}\cos^4\theta\left(1+\tan^2\theta\right)^{3/2}\left(\tan\frac{\pi}{k}\tan\theta+1\right)^3} \qquad \forall\theta \in \left(0, \frac{2\pi}{k}\right).$$

Since $\beta_k'' \geq 0$ in $\left[0, \frac{2\pi}{k}\right]$ and since β_k is symmetric with respect to $\theta = \pi/k$ then β_k achieves the maximum slope as $\theta \to 0^+$ and as $\theta \to \left(\frac{2\pi}{k}\right)^-$, i.e.

$$M_k := \sup_{\theta\in(0,2\pi/k)} |\beta_k'(\theta)| = \lim_{\theta\to 0^+} |\beta_k'(\theta)| = \lim_{\theta\to(2\pi/k)^-} |\beta_k'(\theta)| = \frac{\tan\frac{\pi}{k}}{\cos\frac{\pi}{k}}.$$

Hence, the Lipschitz constant M_k for β_k is uniformly bounded with respect to k. On the other hand, we have

$$\rho_{0k} = 1 \qquad \text{and} \qquad \rho_{1k} = \frac{1}{\cos\frac{\pi}{k}}.$$

Moreover, the uniform outer ball condition for P_k is satisfied by a radius $R > 0$ independent of k. Finally, if we choose $\mu(x) := e^{-1/|x|^2}x$, then there exists $\delta > 0$ independent of k such that

$$\mu\cdot\nu \geq \delta \qquad \text{on } \partial P_k.$$

Then [1, Lemma 4.6] and [1, Theorem 2.2] yield the desired estimate. □

For any $k \geq 3$ let $u_k \in H^2 \cap H_0^1(P_k)$ be a minimizer for $d_1(P_k)$ such that $\|\Delta u_k\|_{L^2(P_k)} = 1$. Then by Lemma 3 we infer that there exists a constant $C > 0$ independent of k such that

$$\|u_k\|_{H^2(P_k)} \leq C\|\Delta u_k\|_{L^2(P_k)} = C \tag{22}$$

and hence if the restriction of u_k to the unit disk D is still denoted by u_k then the sequence $\{u_k\}$ is bounded in $H^2(D)$.

If $\{u_{k_m}\}$ is an arbitrary subsequence then up to extract another subsequence we may assume that $u_{k_m} \rightharpoonup u$ in $H^2(D)$. Our purpose is to prove that

$$\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}} \to \int_{\partial D} u_\nu^2 \tag{23}$$

where ν_{k_m} and ν denotes respectively the outer normals to ∂P_{k_m} and to ∂D.

By Theorem 6.2, Chapter 2 in [20] we obtain

$$\nabla u_{k_m}|_{\partial D} \to \nabla u|_{\partial D} \qquad \text{in } (L^2(\partial D))^2$$

so that

$$\int_{\partial D} (u_{k_m})^2_\nu \to \int_{\partial D} u_\nu^2\,. \tag{24}$$

Therefore in order to prove (23) we only need to prove that

$$\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}} - \int_{\partial D} (u_{k_m})^2_\nu \to 0. \tag{25}$$

In the next lemma we prove that convergence in (25) occurs for the initial sequence $\{u_k\}$.

Lemma 4. *Let $\{u_k\}$ be a sequence of minimizers for $d_1(P_k)$ with $\|\Delta u_k\|_{L^2(P_k)} = 1$. Then*

$$\int_{\partial P_k} (u_k)^2_{\nu_k} - \int_{\partial D} (u_k)^2_\nu \to 0.$$

as $k \to \infty$.

Proof. Let L_k be an arbitrary edge of the polygon P_k and let $S_k \subset \partial D$ be the corresponding arc. Since the set of minimizers for P_k is a 1-dimensional vector space then u_k is invariant under the action of the group of symmetries of the polygon P_k. Therefore we have that

$$\int_{\partial P_k} (u_k)^2_{\nu_k} = k \int_{L_k} (u_k)^2_{\nu_k} \qquad \text{and} \qquad \int_{\partial D} (u_k)^2_\nu = k \int_{S_k} (u_k)^2_\nu. \tag{26}$$

Up to rotations in the plane, it is not restrictive assuming that the edge L_k is horizontal so that $\nu_k \equiv (0, 1)$ on L_k and hence

$$\int_{\partial P_k} (u_k)^2_{\nu_k} = k \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left| \frac{\partial u_k}{\partial x_2}(t, 1) \right|^2 dt. \tag{27}$$

On the other hand, we have

$$\int_{\partial D} (u_k)^2_\nu = k \int_{\frac{\pi}{2}-\frac{\pi}{k}}^{\frac{\pi}{2}+\frac{\pi}{k}} \left| \frac{\partial u_k}{\partial x_1}(\cos\theta, \sin\theta) \cdot \cos\theta + \frac{\partial u_k}{\partial x_2}(\cos\theta, \sin\theta) \cdot \sin\theta \right|^2 d\theta. \tag{28}$$

The rest of the proof is divided in several steps.

Step 1. We prove

$$I_{1k} := \int_{\partial D} (u_k)^2_\nu - k \int_{\frac{\pi}{2}-\frac{\pi}{k}}^{\frac{\pi}{2}+\frac{\pi}{k}} \left| \frac{\partial u_k}{\partial x_2}(\cos\theta, \sin\theta) \right|^2 d\theta \to 0$$

as $k \to \infty$. By (28) and Hölder inequality, we have

$$|I_{1k}| \le k \left[\int_{\frac{\pi}{2}-\frac{\pi}{k}}^{\frac{\pi}{2}+\frac{\pi}{k}} [2|\nabla u_k(\cos\theta, \sin\theta)|]^2 d\theta \right]^{1/2} \left[\int_{\frac{\pi}{2}-\frac{\pi}{k}}^{\frac{\pi}{2}+\frac{\pi}{k}} |\nabla u_k(\cos\theta, \sin\theta)|^2 [\cos^2\theta + (1-\sin\theta)^2] d\theta \right]^{1/2}$$

$$\le 2 \left[\cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right) + 1 - \sin\left(\frac{\pi}{2} - \frac{\pi}{k}\right) \right] \int_{\partial D} |\nabla u_k|^2$$

and by trace inequality

$$\le 2C \left[\cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right) + 1 - \sin\left(\frac{\pi}{2} - \frac{\pi}{k}\right) \right] \|u_k\|^2_{H^2(D)} \to 0$$

as $k \to \infty$ since $\{u_k\}$ is bounded in $H^2(D)$.

After the change of variable $\theta = \arccos t$ by (27) and Step 1, the statement of the lemma is reduced to prove that

$$k \left| \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} \frac{v_k^2(t, \sqrt{1-t^2})}{\sqrt{1-t^2}}\, dt - \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} v_k^2(t, 1) dt \right| \to 0 \tag{29}$$

as $k \to \infty$ where we put $v_k = \frac{\partial u_k}{\partial x_2}$.

Step 2. We prove that

$$I_{2k} := k \left| \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} \frac{v_k^2(t, \sqrt{1-t^2})}{\sqrt{1-t^2}}\, dt - \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} v_k^2(t, \sqrt{1-t^2})\, dt \right| \to 0$$

as $k \to \infty$. We have

$$I_{2k} = k \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} \frac{t^2 v_k^2(t, \sqrt{1-t^2})}{\sqrt{1-t^2}(1+\sqrt{1-t^2})}\, dt \le k \sin^2\frac{\pi}{k} \cdot \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} \frac{v_k^2(t, \sqrt{1-t^2})}{\sqrt{1-t^2}}\, dt$$

$$= \sin^2\frac{\pi}{k} \cdot \int_{\partial D} v_k^2 \le \sin^2\frac{\pi}{k} \cdot \int_{\partial D} |\nabla u_k|^2 \le C \sin^2\frac{\pi}{k} \cdot \|u_k\|_{H^2(D)} \to 0$$

as $k \to \infty$.

Step 3. We show that

$$I_{3k} := k \left| \int_{-\sin\frac{\pi}{k}}^{\sin\frac{\pi}{k}} v_k^2(t, \sqrt{1-t^2})\, dt - \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} v_k^2(t, \sqrt{1-t^2})\, dt \right| \to 0$$

as $k \to \infty$. By the symmetry properties of u_k and Hölder inequality we have

$$I_{3k} = 2k \int_{\sin\frac{\pi}{k}}^{\tan\frac{\pi}{k}} v_k^2(t, \sqrt{1-t^2}) dt \le 2k \left(\tan\frac{\pi}{k} - \sin\frac{\pi}{k}\right)^{1/q'} \left(\int_{\sin\frac{\pi}{k}}^{\tan\frac{\pi}{k}} |v_k(t, \sqrt{1-t^2})|^{2q} dt \right)^{1/q}$$

and by trace inequality

$$\le 2k \left(\tan\frac{\pi}{k} - \sin\frac{\pi}{k}\right)^{1/q'} \|v_k\|^2_{L^{2q}(\partial D)} \le C_q k \left(\tan\frac{\pi}{k} - \sin\frac{\pi}{k}\right)^{1/q'} \|v_k\|^2_{H^1(D)}$$

$$\le C_q k \left(\tan\frac{\pi}{k} - \sin\frac{\pi}{k}\right)^{1/q'} \|u_k\|^2_{H^2(D)} \tag{30}$$

for some constant $C_q > 0$ depending only on q. Since $n = 2$, we may choose $q > 2$ so that (30) tends to zero as $k \to \infty$.

Summarizing, by (29), Step 2 and Step 3, it remains to prove that

$$I_{4k} := k \left| \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} v_k^2(t, \sqrt{1-t^2})\, dt - \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} v_k^2(t, 1)\, dt \right| \to 0 \tag{31}$$

as $k \to \infty$. We proceed as follows.

$$I_{4k} = k \left| \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left[v_k(t,0) + \int_0^{\sqrt{1-t^2}} \frac{\partial v_k}{\partial x_2}(t, x_2) dx_2 \right]^2 dt - \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left[v_k(t,0) + \int_0^1 \frac{\partial v_k}{\partial x_2}(t, x_2) dx_2 \right]^2 dt \right|$$

$$\le 2k \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} |v_k(t,0)| \left(\int_{\sqrt{1-t^2}}^1 \left| \frac{\partial v_k}{\partial x_2}(t, x_2) \right| dx_2 \right) dt$$

$$+ k \int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left| \int_0^1 \frac{\partial v_k}{\partial x_2}(t, x_2) dx_2 + \int_0^{\sqrt{1-t^2}} \frac{\partial v_k}{\partial x_2}(t, x_2) dx_2 \right| \cdot \left| \int_{\sqrt{1-t^2}}^1 \frac{\partial v_k}{\partial x_2}(t, x_2) dx_2 \right| dt$$

and by Hölder inequality

$$\leq 2k\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} |v_k(t,0)|^2 dt\right)^{1/2} \left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left(\int_{\sqrt{1-t^2}}^{1} \left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right| dx_2\right)^2 dt\right)^{1/2}$$

$$+k\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left(2\int_0^1 \left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right| dx_2\right)^2 dt\right)^{1/2} \left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} \left(\int_{\sqrt{1-t^2}}^{1} \left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right| dx_2\right)^2 dt\right)^{1/2}.$$

Since the sequence $\{v_k\}$ is bounded in $H^1(D)$ then by Sobolev embedding $H^1(D) \subset L^\infty(D)$ we have that $\{v_k\}$ is bounded in $L^\infty(D)$ and hence using again Hölder inequality we obtain

$$I_{4k} \leq 2\sqrt{2}k\|v_k\|_{L^\infty(D)}\left(\tan\frac{\pi}{k}\right)^{1/2}\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} (1-\sqrt{1-t^2})\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\,dt\right)^{1/2}$$

$$+2k\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}}\int_0^1\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\,dt\right)^{1/2}\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}} (1-\sqrt{1-t^2})\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\,dt\right)^{1/2}$$

$$\leq Ck\left(\tan\frac{\pi}{k}\right)^{1/2}\left(1-\sqrt{1-\tan^2\frac{\pi}{k}}\right)^{1/2}\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}}\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\,dt\right)^{1/2}$$

$$+2k\|v_k\|_{H^1(P_k)}\left(1-\sqrt{1-\tan^2\frac{\pi}{k}}\right)^{1/2}\left(\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}}\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\,dt\right)^{1/2}.$$

Since $\|v_k\|_{H^1(P_k)} \leq \|u_k\|_{H^2(P_k)}$ is bounded in view of (22) and

$$\left(1-\sqrt{1-\tan^2\frac{\pi}{k}}\right)^{1/2} \sim \frac{\pi}{\sqrt{2}}k^{-1}+o(k^{-1}),$$

then in order to prove that I_{4k} converged to zero as $k\to\infty$ it is sufficient to show that

$$\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}}\left(\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\right)dt \to 0$$

as $k\to\infty$. This follows immediately from the fact that

$$k\int_{-\tan\frac{\pi}{k}}^{\tan\frac{\pi}{k}}\left(\int_{\sqrt{1-t^2}}^{1}\left|\frac{\partial v_k}{\partial x_2}(t,x_2)\right|^2 dx_2\right)dt \leq 3\int_{P_k}\left|\frac{\partial v_k}{\partial x_2}\right|^2 dx \leq 3\|u_k\|_{H^2(P_k)} \leq C$$

for some positive constant independent of k. □

By Theorem 2 we have for any k

$$d_1(P_k)=\delta_1(P_k)=\min_{h\in\mathbf{H}\setminus\{0\}}\frac{\int_{\partial P_k} h^2}{\int_{P_k} h^2} \leq \frac{|\partial P_k|}{|P_k|}$$

and hence

$$\limsup_{k\to\infty} d_1(P_k) \leq \limsup_{k\to\infty}\frac{|\partial P_k|}{|P_k|}=2. \tag{32}$$

In particular we have that the sequence $d_1(P_k)$ is bounded.

By (24), (25) and Lemma 4 we have along a subsequence $\{u_{k_m}\}$ satisfying $u_{k_m} \rightharpoonup u$ in $H^2(D)$ as $m \to \infty$

$$\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}} \to \int_{\partial D} u_\nu^2$$

which yields

$$\int_{\partial D} u_\nu^2 > 0$$

since $d_1(P_{k_m})$ is bounded and

$$\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}} = d_1(P_{k_m})^{-1} \cdot \int_{P_{k_m}} |\Delta u_{k_m}|^2 = d_1(P_{k_m})^{-1}.$$

Finally we have

$$\liminf_{m\to\infty} d_1(P_{k_m}) = \liminf_{m\to\infty} \frac{\int_{P_{k_m}} |\Delta u_{k_m}|^2}{\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}}} \geq \liminf_{m\to\infty} \frac{\int_D |\Delta u_{k_m}|^2}{\int_{\partial P_{k_m}} (u_{k_m})^2_{\nu_{k_m}}} \geq \frac{\int_D |\Delta u|^2}{\int_{\partial D} u_\nu^2} \geq 2. \tag{33}$$

Then (32), (33) imply that along the initial sequence we have

$$\lim_{k\to\infty} d_1(P_k) = 2 = d_1(D)$$

so that the proof of the theorem is complete.

8. Proof of Theorem 7

Let $\delta \in \left(0, 1/(2n^2)\right)$, so that

$$|\theta(x_1) - \theta(x_2)| \leq n^2 \|\theta\|_{C_b^2} |x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2| \qquad \forall x_1, x_2 \in \mathbb{R}^n$$

for any $\theta \in C_b^2(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\theta\|_{C_b^2} < \delta$. Then by the Banach Fixed Point Theorem and the Local Inversion Theorem we infer that the map $I + \theta$ is a diffeomorphism of $\mathbb{R}^n$ of class C^2 for any $\theta \in C_b^2(\mathbb{R}^n; \mathbb{R}^n)$ which satisfies $\|\theta\|_{C_b^2} < \delta$. Put $\Omega = (I+\theta)(\Omega_0)$ and assume that $\overline{w}$ is a minimizer for $d_1(\Omega)$ which satisfies

$$\int_\Omega \left[|D^2\overline{w}| + |\nabla\overline{w}|\right]^2 dx \leq 1, \qquad \int_{\partial\Omega} \overline{w}_\nu^2 dS \leq 1. \tag{34}$$

Let $\overline{u} \in H^2 \cap H_0^1(\Omega_0)$ be given by $\overline{u} = \overline{w} \circ (I+\theta)$ in Ω_0. Then we have

$$\Delta\overline{u} = \sum_{i,j=1}^n \left\{ \left[\sum_{k=1}^n \left(\frac{\partial^2 \overline{w}}{\partial x_k \partial x_j} \circ (I+\theta) \right) \frac{\partial (I+\theta)_k}{\partial x_i} \frac{\partial (I+\theta)_j}{\partial x_i} \right] + \left(\frac{\partial \overline{w}}{\partial x_j} \circ (I+\theta) \right) \frac{\partial^2 (I+\theta)_j}{\partial x_i^2} \right\} \quad \text{in } \Omega_0$$

so that

$$|\Delta\overline{u} - (\Delta\overline{w}) \circ (I+\theta)| \leq C_1 \left[\|\theta\|^2_{C_b^2} \left| (D^2\overline{w}) \circ (I+\theta) \right| + \|\theta\|_{C_b^2} |(\nabla\overline{w}) \circ (I+\theta)| \right] \qquad \text{in } \Omega_0 \tag{35}$$

where C_1 is a positive constant depending only on n. By (35) and Hölder inequality we have

$$\int_{\Omega_0} |\Delta\overline{u}|^2 dx \leq \int_{\Omega_0} |(\Delta\overline{w}) \circ (I+\theta)|^2 dx$$

$$+2C_1\left[\int_{\Omega_0}\left[\|\theta\|_{C_b^2}^2\left|\left(D^2\overline{w}\right)\circ(I+\theta)\right|+\|\theta\|_{C_b^2}\left|(\nabla\overline{w})\circ(I+\theta)\right|\right]^2dx\right]^{1/2}\left[\int_{\Omega_0}\left|(\Delta\overline{w})\circ(I+\theta)\right|^2dx\right]^{1/2}$$

$$+C_1^2\int_{\Omega_0}\left[\|\theta\|_{C_b^2}^2\left|\left(D^2\overline{w}\right)\circ(I+\theta)\right|+\|\theta\|_{C_b^2}\left|(\nabla\overline{w})\circ(I+\theta)\right|\right]^2dx. \tag{36}$$

On the other hand, since the determinant is a locally Lipschitz function with respect to any norm in the space of matrices, for any $\varepsilon\in(0,1)$ we may choose $\delta\in(0,\varepsilon)$ small enough such that

$$\left|\det\left(J(I+\theta)^{-1}(y)\right)-1\right|<\varepsilon\qquad\forall y\in\Omega \tag{37}$$

for any θ with $\|\theta\|_{C_b^2}<\delta$. Combining (34), (36), (37) and using the fact that $\|\theta\|_{C_b^2}<\delta<\varepsilon$ we obtain

$$\int_{\Omega_0}|\Delta\overline{u}|^2\,dx\le\int_\Omega|\Delta\overline{w}|^2\,dy+\varepsilon\int_\Omega|\Delta\overline{w}|^2\,dy$$

$$+2C_1\,(1+\varepsilon)\left(\int_\Omega\left[\varepsilon^2\left|D^2\overline{w}\right|+\varepsilon\left|\nabla\overline{w}\right|\right]^2dy\right)^{1/2}\left(\int_\Omega|\Delta\overline{w}|^2\,dy\right)^{1/2}$$

$$+C_1^2\,(1+\varepsilon)\int_\Omega\left[\varepsilon^2\left|D^2\overline{w}\right|+\varepsilon\left|\nabla\overline{w}\right|\right]^2dy$$

$$\le\int_\Omega|\Delta\overline{w}|^2\,dy+\varepsilon\left(1+4C_1+2C_1^2\right)=\int_\Omega|\Delta\overline{w}|^2\,dy+\varepsilon C_2. \tag{38}$$

On the other hand, after explicit computation of $|\nabla\overline{u}|$ we find

$$\left|\;|\nabla\overline{u}|-|(\nabla\overline{w})\circ(I+\theta)|\;\right|\le C_3\,\|\theta\|_{C_b^2}\,|(\nabla\overline{w})\circ(I+\theta)|\qquad\text{on }\partial\Omega_0$$

where C_3 is a positive constant depending only on n.

Choosing $\delta\le 1/C_3$ so that $1-C_3\,\|\theta\|_{C_b^2}>0$ for $\|\theta\|_{C_b^2}<\delta$, after integration over $\partial\Omega_0$, we obtain

$$\int_{\partial\Omega_0}\overline{u}_\nu^2dS_x\ge\left(1-C_3\,\|\theta\|_{C_b^2}\right)^2\int_{\partial\Omega_0}|(\nabla\overline{w})\circ(I+\theta)|^2\,dS_x. \tag{39}$$

Using a local parametrization for $\partial\Omega_0$ we prove that there exists a positive constant $C_4>0$ depending only on Ω_0 such that

$$\int_{\partial\Omega_0}|(\nabla\overline{w})\circ(I+\theta)|^2\,dS_x\ge\int_{\partial\Omega}|\nabla\overline{w}(y)|^2\,dS_y-C_4\,\|\theta\|_{C_b^2}\ge0\qquad\forall\,\|\theta\|_{C_b^2}<\delta$$

with $\delta>0$ small enough. Inserting this estimate into (39) and using the fact that $\|\theta\|_{C_b^2}<\delta<\varepsilon$ we obtain

$$\int_{\partial\Omega_0}\overline{u}_\nu^2\;dS_x\ge\int_{\partial\Omega}\overline{w}_\nu^2\;dS_y-\varepsilon\left(2C_3+C_4\right). \tag{40}$$

Then, by combining (38) and (40) we infer

$$d_1\left(\Omega_0\right)\le\frac{\displaystyle\int_{\Omega_0}|\Delta\overline{u}|^2\,dx}{\displaystyle\int_{\partial\Omega_0}\overline{u}_\nu^2dS_x}\le\frac{\displaystyle\int_\Omega|\Delta\overline{w}|^2\,dy+\varepsilon C_2}{\displaystyle\int_{\partial\Omega}\overline{w}_\nu^2dS_y-\varepsilon\left(2C_3+C_4\right)}\le d_1\left(\Omega\right)+\varepsilon C_5 \tag{41}$$

for a suitable constant $C_5>0$ depending only on n and Ω_0.

On the other hand, we prove that there exists a constant $C_6 > 0$ depending only on n such that $\| (I+\theta)^{-1} - I\|_{C_b^2} < C_6\delta$ for any $\theta \in C_b^2(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\theta\|_{C_b^2} < \delta$. Reversing the roles of Ω and Ω_0 by (41) we deduce that there exists a constant $C_7 > 0$ depending only on n and Ω_0 such that

$$d_1(\Omega) \leq d_1(\Omega_0) + \varepsilon C_7$$

and hence we obtain

$$|d_1(\Omega) - d_1(\Omega_0)| < \varepsilon \max\{C_5, C_7\}$$

for any $\Omega = (I+\theta)(\Omega_0)$ with $\|\theta\|_{C_b^2} < \delta$ and $\delta = \delta(\varepsilon) > 0$ small enough. This completes the proof of the theorem. □

9. Proof of Theorem 8

Let $B \subset \mathbb{R}^n$ be the unit ball centered at the origin and let $u^0(x) = 1 - |x|^2$ be the unique (up to a constant multiplier) eigenfunction associated to $d_1(B)$ (see [4]). For any $\theta \in C_b^4(\mathbb{R}^n;\mathbb{R}^n)$ let u^θ be the unique positive solution of

$$\begin{cases} \Delta^2 u^\theta = 0 & \text{in } (I+\theta)(B) \\ u^\theta = 0 & \text{on } \partial((I+\theta)(B)) \\ \Delta u^\theta - d_1((I+\theta)(B))\, u_\nu^\theta = 0 & \text{on } \partial((I+\theta)(B)) \end{cases} \tag{42}$$

such that

$$\|\Delta u^\theta\|_{L^2((I+\theta)(B))} = 2n\sqrt{|B|} = \|\Delta u^0\|_{L^2(B)}. \tag{43}$$

Consider the functional $\theta \mapsto d_1(\theta)$ defined by

$$d_1(\theta) = d_1((I+\theta)(B)) = \frac{\displaystyle\int_{(I+\theta)(B)} |\Delta u^\theta|^2 dx}{\displaystyle\int_{\partial((I+\theta)(B))} |u^\theta|_\nu^2 \, dS} \qquad \forall \theta \in C_b^4(\mathbb{R}^n;\mathbb{R}^n). \tag{44}$$

In the first part of this section we prove that the functional $\theta \mapsto d_1(\theta)$ is differentiable in a neighbourhood of $\theta = 0$.

Let

$$\Theta = \left\{ \theta \in C_b^4(\mathbb{R}^n;\mathbb{R}^n) : \|\theta\|_{C_b^4} < \frac{1}{2n^2} \right\}$$

so that $I+\theta$ is a C^4-diffeomorphism of $\mathbb{R}^n$ for any $\theta \in \Theta$.

Let $v^\theta = u^\theta \circ (I+\theta)$ so that $v^\theta \in H^2 \cap H_0^1(B)$ for any $\theta \in \Theta$. Since $\partial((I+\theta)(B)) \in C^4$ and since the Steklov boundary conditions satisfy the complementing conditions (see Lemma 15 in [4]) by elliptic regularity [2] we know that $u^\theta \in H^4((I+\theta)(B))$ and, in turn, also $v^\theta \in H^4(B)$ for any $\theta \in \Theta$.

Consider the transposed inverse matrix of the Jacobian of the map $I+\theta$,

$$M(\theta) = [M_{ij}(\theta)] = \left([J(I+\theta)]^{-1} \right)^T \qquad \forall \theta \in \Theta$$

and define the linear operator L_θ by

$$L_\theta u = \sum_{i,j,k=1}^n M_{ij}(\theta) \frac{\partial}{\partial x_j}\left(M_{ik}(\theta) \frac{\partial u}{\partial x_k} \right) = \Delta\left(u \circ (I+\theta)^{-1} \right) \circ (I+\theta) \quad \text{in } B \tag{45}$$

for any smooth function u defined on B. Then the function v^θ solves the problem

$$\begin{cases} L_\theta^2 v^\theta = 0 & \text{in } B \\ v^\theta = 0 & \text{on } \partial B \\ L_\theta v^\theta - d_1(\theta) \sum_{i,j=1}^{n} \frac{\partial v^\theta}{\partial x_j} M_{ij}(\theta) \left(\nu_i^\theta \circ (I+\theta)\right) = 0 & \text{on } \partial\left((I+\theta)(B)\right) \end{cases} \tag{46}$$

where $\nu^\theta = \left(\nu_1^\theta, ..., \nu_n^\theta\right)$ is the unit normal vector to the boundary $\partial\left((I+\theta)(B)\right)$. Define the map

$$F : \Theta \times \mathbb{R} \times \left(H^4 \cap H_0^1(B)\right) \to L^2(B) \times H^{3/2}(\partial B) \times \mathbb{R}$$

by

$$F(\theta, d, v) = \left(L_\theta^2 v, L_\theta v - d \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j} M_{ij}(\theta) \left(\nu_i^\theta \circ (I+\theta)\right), \frac{1}{2} \int_B |\Delta v|^2 \, dx \right).$$

Note that the map $\theta \mapsto M_{ij}(\theta)$ is of class C^1 from Θ into $C_b^3(\mathbb{R}^n)$ for any $i, j \in \{1, ..., n\}$ where

$$C_b^k(\mathbb{R}^n) = \left\{ u : \partial^\alpha u \in C^0 \cap L^\infty(\mathbb{R}^n) \text{ for any } 0 \le |\alpha| \le k \right\},$$

see Section 1.3 in [22] for more details. This implies that the map F is of class C^1 in $\Theta \times \mathbb{R} \times \left(H^4 \cap H_0^1(B)\right)$.

Finally define the set

$$\mathcal{Z} = \left\{ (\theta, d, v) \in \Theta \times \mathbb{R} \times \left(H^4 \cap H_0^1(B)\right) : F(\theta, d, v) = \left(0, 0, 2n^2 |B|\right) \right\}$$

so that $\left(0, d_1(B), u^0\right) \in \mathcal{Z}$. By means of the Implicit Function Theorem we prove the following

Lemma 5. *There exist a neighbourhood U of $\theta = 0$ in $C_b^4(\mathbb{R}^n; \mathbb{R}^n)$, a neighbourhood V of $\left(d_1(B), u^0\right)$ in $\mathbb{R} \times \left(H^4 \cap H_0^1(B)\right)$ and a map $\Lambda : U \to V$ of class C^1 such that $(d, v) = \Lambda(\theta)$ for any $(\theta, d, v) \in \mathcal{Z} \cap (U \times V)$. Moreover $\Lambda(\theta = 0) = \left(d_1(B), u^0\right)$ with $u^0(x) = 1 - |x|^2$.*

Proof. The partial variation of the map F with respect to the the pair (d, v) takes the form

$$F'_{(d,v)}(\theta_0, d_0, v_0)[(d, v)] =$$

$$= \begin{pmatrix} L_{\theta_0}^2 v \\ L_{\theta_0} v - d \sum_{i,j=1}^{n} \frac{\partial v_0}{\partial x_j} M_{ij}(\theta_0) \left(\nu_i^{\theta_0} \circ (I+\theta_0)\right) - d_0 \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j} M_{ij}(\theta_0) \left(\nu_i^{\theta_0} \circ (I+\theta_0)\right) \\ \int_B \Delta v_0 \Delta v \, dx \end{pmatrix}.$$

We show that $F'_{(d,v)}\left(0,d_1(B),u^0\right):\mathbb{R}\times\left(H^4\cap H_0^1(B)\right)\to L^2(B)\times H^{3/2}(\partial B)\times\mathbb{R}$ is an isomorphism. This is equivalent to show that for any $f\in L^2(B)$, $g\in H^{3/2}(\partial B)$, $\alpha\in\mathbb{R}$ the problem

$$\text{(47)}\qquad \begin{cases}\Delta^2 v=f & \text{in } B\\ v=0 & \text{on } \partial B\\ \Delta v-d_1(B)\,v_\nu-du_\nu^0=g & \text{on } \partial B\end{cases}$$

$$\text{(48)}\qquad \int_B \Delta u^0\Delta v\;dx=\alpha$$

admits a unique solution $(d,v)\in\mathbb{R}\times\left(H^4\cap H_0^1(B)\right)$. We prove this statement in two steps.

Existence of a solution of (47)-(48). We start by looking for a solution $(d,v)\in\mathbb{R}\times H^2\cap H_0^1(B)$ of (47), i.e. a pair (d,v) which satisfies

$$\text{(49)}\qquad \int_B \Delta v\Delta w\;dx-d_1(B)\int_{\partial B} v_\nu w_\nu\;dS=\int_B fw\;dx+\int_{\partial B}\left(g+du_\nu^0\right)w_\nu\;dS\qquad \forall w\in H^2\cap H_0^1(B).$$

By Lemma 2 we deduce that (49) admits a solution if and only if the following condition holds

$$\text{(50)}\qquad \int_B fu^0\;dx+\int_{\partial B}\left(g+du_\nu^0\right)u_\nu^0\;dS=0.$$

Therefore if we choose $d\in\mathbb{R}$ such that (50) holds true then there exists a solution $v\in H^2\cap H_0^1(B)$ of (49) and by elliptic regularity [2] it follows that $v\in H^4(B)$. In general this function v does not satisfy (48) but it is worth noting that if v solves (49) then for any $\lambda\in\mathbb{R}$ the function $v_\lambda=v+\lambda u^0$ is still a solution of (49). It is now sufficient to choose λ such that (48) is satisfied. With this choice of λ the corresponding function $v_\lambda\in H^4\cap H_0^1(B)$ solves (47)-(48).

Uniqueness for (47)-(48). In order to prove uniqueness for (47)-(48) it is sufficient to prove that the associated homogeneous problem

$$\text{(51)}\qquad \begin{cases}\Delta^2 v=0 & \text{in } B\\ v=0 & \text{on } \partial B\\ \Delta v-d_1(B)\,v_\nu-du_\nu^0=0 & \text{on } \partial B\end{cases}$$

$$\text{(52)}\qquad \int_B \Delta u^0\Delta v\;dx=0$$

admits only the trivial solution $(0,0)\in\mathbb{R}\times\left(H^4\cap H_0^1(B)\right)$. The corresponding variational formulation for (51) is

$$\text{(53)}\qquad \int_B \Delta v\Delta w\;dx-d_1(B)\int_{\partial B} v_\nu w_\nu\;dS=d\int_{\partial B} u_\nu^0 w_\nu\;dS\qquad \forall w\in H^2\cap H_0^1(B).$$

Choosing $w=v$ in (53), by (52) and using the fact that u^0 is an eigenfunction of $d_1(B)$, we obtain

$$\begin{aligned}\int_B |\Delta v|^2\,dx&=d_1(B)\int_{\partial B} v_\nu^2 dS+d\int_{\partial B} u_\nu^0 v_\nu\;dS\\ &=d_1(B)\int_{\partial B} v_\nu^2 dS+\frac{d}{d_1(B)}\int_B \Delta u^0\Delta v\;dx=d_1(B)\int_{\partial B} v_\nu^2 dS.\end{aligned}$$

Therefore, if we assume by contradiction that v is not identically equal to zero then we infer that $v\in H^2\cap H_0^1(B)$ is a minimizer for (3) and since $d_1(B)$ is a simple eigenvalue then v coincides with u^0 up to a constant multiplier. This contradicts (52) and hence $v\equiv 0$.

Choosing $w=u^0$ in (53) and using the fact that $v\equiv 0$ we conclude that $d=0$. This completes the proof of uniqueness.

We may conclude that $F'_{(d,v)}\left(0,d_1(B),u^0\right)$ is an isomorphism and since the map F is of class C^1 in $\Theta\times\mathbb{R}\times\left(H^4\cap H^1_0(B)\right)$. Hence, the statement of the lemma follows from the Implicit Function Theorem. □

In the next statement we show that if $\Lambda(\theta)=(d(\theta),v(\theta))$ is the map found in Lemma 5 then $d(\theta)=d_1(\theta)$ and $v(\theta)=u^\theta\circ(I+\theta)$ with u^θ as in (42). We recall that $d_1(\theta)$ is defined in (44).

Lemma 6. *Let $\Lambda:U\to V$ the map found in Lemma 5 and let $d(\theta)$ be such that $\Lambda(\theta)=(d(\theta),v(\theta))$ for any $\theta\in U$. Then there exists a neighbourhood $\widetilde{U}\subset U$ of $\theta=0$ with respect to the topology of $C^4_b(\mathbb{R}^n;\mathbb{R}^n)$ such that $d(\theta)=d_1(\theta)$ and $v(\theta)=u^\theta\circ(I+\theta)$ for any $\theta\in\widetilde{U}$ with u^θ as in (42).*

Proof. Suppose by contradiction that there exists a sequence $\{\theta_r\}_{r\in\mathbb{N}}\subset U$ such that $\theta_r\to 0$ in $C^4_b(\mathbb{R}^n;\mathbb{R}^n)$ and $d(\theta_r)\neq d_1(\theta_r)$ for any $r\in\mathbb{N}$. This means that for any fixed $r\in\mathbb{N}$, $d(\theta_r)$ coincides with some eigenvalue $d_{i_r}(\theta_r)$ of (1) with $i_r\neq 1$.

Let u^θ as in (42). By elliptic regularity estimates [2] we deduce that there exists C_θ such that $\left\|u^\theta\right\|_{H^4((I+\theta)(B))}\le C_\theta$. Since the map $\theta\mapsto d_1(\theta)$ is continuous in Θ (see Proposition 7) and $\theta\in\Theta$, the elliptic regularity estimates up to the boundary may be done in such a way that the constant C_θ can be chosen independent of $\theta\in\Theta$. Therefore there exist $C>0$ such that

$$\|u^\theta\|_{H^4((I+\theta)(B))}\le C\qquad\forall\theta\in\Theta.$$

Then one can prove by computing all the derivatives of a composition up to the fourth order that there exists a positive constant still denoted by C such that

$$\|\psi_1^\theta\|_{H^4(B)}\le C\qquad\forall\theta\in\Theta$$

with $\psi_1^\theta=u^\theta\circ(I+\theta)$. Therefore up to subsequences we may assume that there exists $\psi\in H^4(B)$ such that $\psi_1^{\theta_r}\rightharpoonup\psi$ weakly in $H^4(B)$ and by compact embedding we also have $\psi_1^{\theta_r}\to\psi$ strongly in $H^2\cap H^1_0(B)$. Therefore since $M_{ij}(\theta_r)\to\delta_{ij}$ in $C^3_b(\mathbb{R}^n)$ for any $i,j\in\{1,..,n\}$ we also have

$$4n^2|B|=\lim_{r\to\infty}\int_{(I+\theta)(B)}|\Delta u^{\theta_r}|^2dx=\lim_{r\to\infty}\int_B|L_{\theta_r}\psi_1^{\theta_r}|^2\left|\det J(I+\theta_r)\right|dx=\int_B|\Delta\psi|^2\,dx$$

which shows that $\psi\neq 0$.

On the other hand, $\psi_1^{\theta_r}$ solves (46) with θ_r in place of θ and hence we have

$$\sum_{i,j,k=1}^n\int_B L_{\theta_r}\psi_1^{\theta_r}\frac{\partial}{\partial x_k}\left(M_{ik}(\theta_r)\frac{\partial}{\partial x_j}\left(M_{ij}(\theta_r)w\right)\right)dx$$

(54)

$$-\sum_{i,j,k,l,m=1}^n\int_{\partial B}M_{ik}(\theta_r)\frac{\partial}{\partial x_j}\left(M_{ij}(\theta_r)w\right)\nu_k d_1(\theta_r)M_{lm}(\theta_r)\left(\nu_l^{\theta_r}\circ(I+\theta_r)\right)\frac{\partial\psi_1^{\theta_r}}{\partial x_m}dS=0$$

for any $w\in H^2\cap H^1_0(B)$.

Using the following convergences $\psi_1^{\theta_r}\to\psi$ in $H^2\cap H^1_0(B)$, $M_{ij}(\theta_r)\to\delta_{ij}$ in $C^3_b(\mathbb{R}^n)$ for any $i,j\in\{1,..,n\}$, $\nu_i^{\theta_r}\circ(I+\theta_r)\to\nu_i$ in $L^\infty(\partial B)$ for any $i\in\{1,..,n\}$ and $d_1(\theta_r)\to d_1(B)$ in view of Proposition 7, passing to the limit in (54) we obtain

$$\int_B\Delta\psi\Delta w\,dx-d_1(B)\int_{\partial B}\psi_\nu w_\nu\,dS=0\qquad\qquad\forall w\in H^2\cap H^1_0(\Omega)(B).$$

Since for any $r \in \mathbb{N}$, $d(\theta_r) \neq d_1(\theta_r)$ and since by (45)-(46) the function $v(\theta_r) \circ (I+\theta_r)^{-1}$ solves (42) in $(I+\theta_r)(B)$ with $d(\theta_r)$ in place of $d_1(\theta_r)$ then by the orthogonality result in Theorem 1.1 in [8] we have

$$0 = \lim_{r\to\infty} \int_{(I+\theta_r)(B)} \Delta u^{\theta_r} \Delta \left(v(\theta_r) \circ (I+\theta_r)^{-1} \right) \, dx$$
$$= \lim_{r\to\infty} \int_B L_{\theta_r} \psi_1^{\theta_r} L_{\theta_r} (v(\theta_r)) \left|\det J (I+\theta_r)\right| dx = \int_B \Delta\psi \Delta u^0 dx$$

and this is a contradiction since ψ and u^0 are nontrivial eigenfunctions of the simple eigenvalue $d_1(B)$ (see Theorem 1 in [4]). □

In order to compute the first variation of $d_1(\theta)$ with respect to θ we introduce the functionals

$$J(\theta) = \int_{(I+\theta)(B)} |\Delta u^\theta|^2 dx, \qquad K(\theta) = \int_{\partial((I+\theta)(B))} |\nabla u^\theta|^2 dS = \int_{\partial((I+\theta)(B))} |u^\theta_\nu|^2 dS$$

so that

$$d_1(\theta) = \frac{J(\theta)}{K(\theta)}.$$

In the next lemma we prove that the functionals J and K are of class C^1 in a neighbourhood of $\theta = 0$.

Lemma 7. *There exists a neighbourhood U of $\theta = 0$ in $C^4_b(\mathbb{R}^n;\mathbb{R}^n)$ such that the functionals J and K are of class C^1 in U. Moreover the directional derivatives of J, K at $\theta = 0$ in the direction $\tau \in C^4_b(\mathbb{R}^n;\mathbb{R}^n)$ take the form*

$$\frac{\partial J(\theta)}{\partial\tau}(0) = \left\langle J'(\theta)_{|\theta=0}, \tau \right\rangle = -4n \int_{\partial B} x\nabla \left(\frac{\partial u^\theta}{\partial\tau}_{|\theta=0} \right) dS + 4n^2 \int_{\partial B} x\tau \, dS \tag{55}$$

$$\frac{\partial K(\theta)}{\partial\tau}(0) = \left\langle K'(\theta)_{|\theta=0}, \tau \right\rangle = -4 \int_{\partial B} x\nabla \left(\frac{\partial u^\theta}{\partial\tau}_{|\theta=0} \right) dS + (4n+4) \int_{\partial B} x\tau \, dS. \tag{56}$$

Proof. We start with the functional J. In view of Lemmas 5, 6 we know that there exists a neighbourhood U of $\theta = 0$ such that the map $\theta \mapsto v(\theta)$ is of class C^1 from U into $H^4 \cap H^1_0(B)$ so that the assumptions (3.2), (3.6), (3.8) and (3.9) of Theorem 3.3 in [22] hold true with $m = 4$ and $p = 2$. The assumption (3.14) of Theorem 3.3 in [22] is also true in view of Theorem 3.4 in [22]. Therefore all the assumptions of Theorem 3.3 are satisfied in any point $\theta \in U$ so that J is of class C^1 in U.

Let us recall that from [4] we know that

$$d_1(B) = n\,, \qquad u^0(x) = 1 - |x|^2 \quad \text{in} \quad B. \tag{57}$$

Then, by (3.15) in [22] we infer

$$\frac{\partial J(\theta)}{\partial\tau}(0) = \int_B 2\Delta u^0 \Delta \left(\frac{\partial u^\theta}{\partial\tau}_{|\theta=0} \right) dx + \int_{\partial B} \left|\Delta u^0\right|^2 x\tau \, dS$$
$$= -4n \int_B \Delta \left(\frac{\partial u^\theta}{\partial\tau}_{|\theta=0} \right) dx + 4n^2 \int_{\partial B} x\tau \, dS.$$

By Green formula we immediately obtain (55).

Consider now the functional K. Using again Lemmas 5, 6 we deduce the assumptions (5.1)-(5.6) of Theorem 5.1 in [22] hold true with $m = 4$ and $p = 2$. The assumption (5.7) of Theorem 5.1 in [22] follows from Theorem 3.4 in [22]. Therefore we may apply Theorem 5.1 in [22] in any point $\theta \in U$ so that K is of class C^1 in U. The formula (56) for directional derivatives of K at $\theta = 0$ follows immediately from (5.8) in [22] and (57). □

We may now complete the proof of Theorem 8.

From Lemma 7 we deduce that the functional $\theta \mapsto d_1(\theta)$ is of class C^1 in a neighbourhood U of $\theta = 0$. Moreover by (55)-(56) and taking into account that $d_1(\theta = 0) = n$ (see (57)) we deduce that the directional derivative of $d_1(\theta)$ at $\theta = 0$ in the direction $\tau \in C_b^4(\mathbb{R}^n;\mathbb{R}^n)$ takes the form

$$
\begin{aligned}
\frac{\partial d_1(\theta)}{\partial \tau}(0) &= \left\langle d_1'(\theta)_{|\theta=0}, \tau \right\rangle = K(0)^{-2}\left(\frac{\partial J(\theta)}{\partial \tau}_{|\theta=0} K(0) - J(0)\frac{\partial K(\theta)}{\partial \tau}_{|\theta=0}\right) \\
&= \left(\int_{\partial B} (u^0)_\nu^2\, dS\right)^{-1}\left[-4n\int_{\partial B} x\nabla\left(\frac{\partial u^\theta}{\partial \tau}_{|\theta=0}\right) dS + 4n^2\int_{\partial B} x\tau\ dS\right] \\
&\quad - \left(\int_{\partial B} (u^0)_\nu^2\, dS\right)^{-1} d_1(B)\left[-4\int_{\partial B} x\nabla\left(\frac{\partial u^\theta}{\partial \tau}_{|\theta=0}\right) dS + (4n+4)\int_{\partial B} x\tau\ dS\right] \\
&= -\frac{n}{|\partial B|}\int_{\partial B} x\tau\ dS = -\frac{1}{|B|}\int_{\partial B} x\tau\ dS. \qquad (58)
\end{aligned}
$$

Introduce now the volume functional $V(\theta)$ defined by

$$
V(\theta) = |(I+\theta)(B)| = \int_{(I+\theta)(B)} dx \qquad \forall \theta \in C_b^4(\mathbb{R}^n;\mathbb{R}^n).
$$

Then by Theorem 3.3 in [22] we infer that

$$
\frac{\partial V(\theta)}{\partial \tau}(0) = \left\langle V'(\theta)_{|\theta=0}, \tau\right\rangle = \int_{\partial B} x\tau\ dS \qquad \forall \tau \in C_b^4(\mathbb{R}^n;\mathbb{R}^n).
$$

Let $\gamma \in C^1\left([0,1]; C_b^4(\mathbb{R}^n;\mathbb{R}^n)\right)$ be such that

$$
\gamma(0) = I \quad \text{and} \quad |\gamma(t)(B)| = |B| \qquad \forall t \in [0,1],
$$

then

$$
\begin{aligned}
&\frac{d}{dt} d_1(\gamma(t)(B))_{|t=0} = \frac{d}{dt} d_1(\gamma(t) - I)_{|t=0} = \left\langle d_1'(\gamma(t)-I)_{|t=0}, \gamma'(0)\right\rangle \\
&= -\frac{1}{|B|}\int_{\partial B} x\gamma'(0)\ dS = -\frac{1}{|B|}\left\langle V'(\gamma(t)-I)_{|t=0}, \gamma'(0)\right\rangle = -\frac{1}{|B|}\frac{d}{dt}V(\gamma(t)-I)_{|t=0} \\
&= -\frac{1}{|B|}\frac{d}{dt}|\gamma(t)(B)|_{|t=0} = 0.
\end{aligned}
$$

This completes the proof. □

References

[1] V. Adolfsson, *L^2-integrability of second order derivatives for Poisson's equation in nonsmooth domains*, Math. Scand. 70, 1992, 146-160

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math. 12, 1959, 623–727

[3] I. Babuška, *Stabilität des Definitionsgebietes mit Rücksicht auf grundlegende Probleme der Theorie der partiellen Differentialgleichungen auch im Zusammenhang mit der Elastizitätstheorie. I, II*, Czechoslovak Math. J. 11 (86), 1961, 76–105, 165–203

[4] E. Berchio, F. Gazzola, E. Mitidieri, *Positivity preserving property for a class of biharmonic elliptic problems*, J. Diff. Eq. 229, 2006, 1–23

[5] E. Berchio, F. Gazzola, T. Weth, *Critical growth biharmonic elliptic problems under Steklov-type boundary conditions*, Adv. Diff. Eq. 12, 2007, 381–406

[6] D. Bucur, G. Buttazzo, *Variational methods in shape optimization problems. Progress in Nonlinear Differential Equations and their Applications, 65.*, Birkhäuser Boston, Inc., Boston, MA, 2005

[7] G. Faber, *Beweis, dass unter allen homogenen membranen von gleicher fläche und gleicher spannung die kreisförmige den tiefsten grundton gibt*, Sitz. Ber. Bayer. Akad. Wiss. 1923, 169–172

[8] A. Ferrero, F. Gazzola, T. Weth, *On a fourth order Steklov eigenvalue problem*, Analysis 25, 2005, 315–332

[9] G. Fichera, *Su un principio di dualità per talune formole di maggiorazione relative alle equazioni differenziali*, Atti Accad. Naz. Lincei 19, 1955, 411–418

[10] F. Gazzola, G. Sweers, *On positivity for the biharmonic operator under Steklov boundary conditions*, to appear in Arch. Rat. Mech. Anal.

[11] D.S. Jerison, C.E. Kenig, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc. 4, 1981, 203–207

[12] D.S. Jerison, C.E. Kenig, *Boundary value problems on Lipschitz domains*, Stud. Part. Diff. Eq. 23, 1982, 1–68

[13] E. Krahn, *Über eine von Rayleigh formulierte minimaleigenschaft des kreises*, Math. Ann. 94, 1925, 97–100

[14] E. Krahn, *Über minimaleigenschaften der kugel in drei und mehr dimensionen*, Acta Comm. Univ. Dorpat. A9, 1926, 1–44

[15] J.R. Kuttler, *Remarks on a Stekloff eigenvalue problem*, SIAM J. Numer. Anal. 9, 1972, 1–5

[16] J.R. Kuttler, *Dirichlet eigenvalues*, SIAM J. Numer. Anal. 16, 1979, 332–338

[17] J.R. Kuttler, V.G. Sigillito, *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl. 23, 1968, 148–160

[18] J.R. Kuttler, V.G. Sigillito, *Estimating eigenvalues with a posteriori/a priori inequalities*, Research Notes in Mathematics, Pitman Advanced Publishing Program 1985

[19] S.A. Nazarov and G. Sweers, *A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners*, J. Diff. Eq. 233, 2007, 151–180

[20] J. Necas, *Les méthodes directes en théorie des équations elliptiques*, Masson et C^{ie} 1967

[21] L.E. Payne, *Some isoperimetric inequalities for harmonic functions*, SIAM J. Math. Anal. 1, 1970, 354–359

[22] J. Simon, *Differentiation with respect to the domain in boundary value problems* Numer. Funct. Anal. Optimiz. 2, 1980, 649–687

[23] J. Smith, *The coupled equation approach to the numerical solution of the biharmonic equation by finite differences, I*, SIAM J. Numer. Anal. 5, 1968, 323–339

[24] J. Smith, *The coupled equation approach to the numerical solution of the biharmonic equation by finite differences, II*, SIAM J. Numer. Anal. 7, 1970, 104–111

[25] W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole Norm. Sup. 19, 1902, 455–490