Bending and stretching energies in a rectangular plate modeling suspension bridges

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Abstract

A rectangular plate modeling the roadway of a suspension bridge is considered. Both the contributions of the bending and stretching energies are analyzed. The latter plays an important role due to the presence of the free edges. A linear model is first considered; in this case, separation of variables is used to determine explicitly the deformation of the plate in terms of the vertical load. Moreover, the same method allows us to study the spectrum of the linear operator and the least eigenvalue. Then the stretching energy is introduced without linearization and the equation becomes quasilinear; the nonlinear term also affects the boundary conditions. We consider two quasilinear models; the *surface increment model* (SIM) in which the stretching energy is proportional to the increment of surface and a *nonlocal model* (NLM) introduced by Berger in the 50's (see [3]). The (SIM) and the (NLM) are studied in detail. According to the strength of prestressing we prove the existence of multiple equilibrium positions.
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1 Introduction

Consider a rectangular plate hinged at two opposite edges and free on the remaining two edges. We have in mind is a suspension bridge and our purpose is to study several different models to describe its behavior. We view the roadway of the bridge as a long narrow rectangular thin plate, hinged on its short edges where the bridge is supported by the ground, and free on its long edges. Let L denote its length and 2ℓ denote its width; a realistic assumption is that $2\ell \cong \frac{L}{100}$. For simplicity, we take $L = \pi$ so that, in the sequel,

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 \,. \tag{1}$$

This model is considered in [8] where the analysis of the bending energy of the plate leads to a fourth order elliptic equation. However, motivated by the presence of free parts of the boundary, the stretching energy was neglected. A more accurate analysis would have to take account of the stretching energy. From a mathematical point of view, one may notice that $H^2 \subset H^1$ and that the H^2-norm bounds the H^1-norm; whence, the stretching energy may be considered as a "lower order term" when compared with the bending energy. But, as we shall see, the former plays an important role in the model and a deep motivation to introduce the stretching energy comes from structural engineering and physics. Concrete is weakly elastic and heavy loads can produce cracks. On the other hand, metals are more elastic and react to loads by bending. For this reason, prestressed concrete structures have been conceived. According to [19, p.28], the father of prestressed concrete bridges is the French engineer Eugène Freyssinet (1879-1962) and these bridges were first built some 75 years ago. Prestressing metal tendons (generally of high tensile steel) are used to provide a clamping load which produces a compressive stress that balances the tensile stress that the concrete compression member would otherwise experience due to a load. Prestressed concrete is obtained by casting concrete around tensioned tendons; this method produces a winning interaction between tendons and concrete. It protects the tendons from corrosion and allows for direct transfer of tension. The concrete adheres and bonds to the bars, and when the tension is released it is transferred to the concrete as compression by static friction. This technique creates an "elasticized concrete" and explains on the one hand why stretching is negligible when computing the total energy of the plate Ω, and on the other hand why it should not be neglected if the purpose is to analyze a more precise model.

In the present paper we study the plate model by considering also the stretching energy. First we consider a linear model and we prove that the problem is well-posed. With a separation of variables method we also determine the explicit solution and this allows to quantify the cross behavior of the plate, a phenomena clearly visible in the Tacoma Narrows Bridge collapse [23]. Next, we analyze the competing effect between bending and stretching, which occurs in prestressed structures; this brings us to study an eigenvalue problem and its spectrum. With these results, the linear theory seems to be sufficiently clear. However, in recent years, the nonlinear structural behavior of suspension bridges has been uncovered, see e.g. [4, 10, 15, 22]. Therefore, a linear model may not be sufficiently accurate to describe the static behavior of bridges, especially for large deflections as the ones visible in the video [23]. Although mathematical models in nonlinear elasticity are fairly complicated it seems that their use is unavoidable. In an important paper, dated some decades ago, Gurtin [12] showed the necessity of nonlinear models in elasticity and concludes his work by writing

Our discussion demonstrates why this theory is far more difficult than most nonlinear theories of mathematical physics. It is hoped that these notes will convince analysts that nonlinear elasticity is a fertile field in which to work.

Since fully nonlinear plate equations appear intractable, and since linear equations fail to highlight important phenomena, a first step should be to study models having some nonlinearity only in the lower order terms. This appears to be a good compromise between too poor linear models and too complicated fully nonlinear models. This compromise is quite common in elasticity, see e.g. the book by Ciarlet [6, p.322] who describes the method of asymptotic expansions for the thickness ε of a plate as a "partial linearization"

in that a system of quasilinear partial differential equations, i.e., with nonlinearities in the higher order terms, is replaced as $\varepsilon \to 0$ by a system of semilinear partial differential equations, i.e., with nonlinearities only in the lower order terms.

In this paper we make one further step towards fully nonlinear models. Instead of a semilinear model, where the nonlinearities merely appear in the zero order term of the differential equation, we consider two quasilinear models with nonlinearities involving derivatives of the unknown function (the vertical displacement of the plate). The Euler-Lagrange equation contains second order nonlinear differential operators while the highest order operator (fourth order) remains linear. We first consider the stretching energy as a multiple of the increment of the surface and not just its first order asymptotic approximation which is usually employed to describe small displacements of the plate. Then we consider a nonlocal quasilinear model (NLM) going back to Berger [3] which may be seen as a second order approximation of the stretching energy. For both these models we derive a quasilinear Euler-Lagrange equation and we prove existence and multiplicity results depending on the strength of prestressing.

This paper is organized as follows. In Section 2 we recall the classical model for the energy of an elastic plate. In Section 3 we study the linearized model and we state the corresponding results: well-posedness (Theorem 1), explicit form of the solution and behavior of the cross bending (Theorem 2), analysis of the least eigenvalue and of the whole spectrum of the linearized bending-stretching competition (Theorem 4). In Section 4 we consider the quasilinear (SIM): after deriving the Euler-Lagrange equation (30) from the minimization of the energy, we prove existence and multiplicity results for both the homogeneous problem (Theorem 5) and the inhomogeneous problem (Theorem 6). In Section 5 we introduce the (NLM) by adapting the Berger model to our partially hinged plate: we derive the Euler-Lagrange equation (38) from the minimization of the energy, then we prove existence and multiplicity results for both the homogeneous problem (Theorem 7) and the inhomogeneous problem (Theorem 8). Finally, Sections 6-11 are devoted to the proofs of the results.

2 A model for a partially hinged plate

Stretching occurs when the horizontal position of the plate is fixed on the boundary $\partial\Omega$ and a deformation of Ω yields a variation of its surface. In our case, the plate is fixed on the two short edges and a deformation of the plate necessarily yields a variation of its surface. Von Kármán [26] assumes that the elastic force is proportional to the increment of surface in such a way that the stretching energy for the plate, whose vertical deflection is u, reads

$$\mathbb{E}_S(u) = \int_\Omega \left(\sqrt{1+|\nabla u|^2} - 1\right)\, dxdy. \tag{2}$$

For small deformations u, the asymptotic expansion $(\sqrt{1+|\nabla u|^2} - 1) \sim |\nabla u|^2/2$ leads to the Dirichlet integral

$$\mathbb{E}_S(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dxdy. \tag{3}$$

The classical bending theory for an elastic beam goes back to Bernoulli and Euler. This theory was extended to plates by Kirchhoff [13] and Love [17]. The bending energy of the plate Ω involves curvatures of the surface, that is, second order derivatives. Let κ_1 and κ_2 denote the principal curvatures of the graph of the function u representing the deformation of the plate. Then, according to [9, 13] (see also [11, Section 1.1.2]), the bending energy of the deformed plate Ω is given by

$$\mathbb{E}_B(u) = \frac{Eh^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2\right) dxdy \tag{4}$$

where h denotes the thickness of the plate, σ the Poisson ratio defined by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ and E the Young modulus defined by $E = 2\mu(1+\sigma)$, with the Lamé constants $\lambda, \mu > 0$ depending on the material. Then

$$0 < \sigma < \frac{1}{2}. \tag{5}$$

For metals we have $\sigma \approx 0.3$, see [17, p.105].

For small deformations u the terms in (4) are approximations of the second order derivatives of u. More precisely, for small deflections u, one has

$$(\kappa_1+\kappa_2)^2 \approx (\Delta u)^2\,, \quad \kappa_1\kappa_2 \approx \det(D^2u) = u_{xx}u_{yy} - u_{xy}^2\,,$$

and therefore

$$\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2 \approx \frac{1}{2}(\Delta u)^2 + (\sigma-1)\det(D^2u).$$

This leads to a bending energy given by

$$\mathbb{E}_B(u) = \frac{Eh^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma-1)\det(D^2u)\right) dxdy.$$

Whence, up to a constant multiplier, the bending energy of the plate is given by

$$\mathbb{E}_B(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right) dxdy\,. \tag{6}$$

Note that, by (5), the quadratic part of the functional (6) is positive.

Summarizing, by recalling both (3) and (6), the total elastic energy of the plate is given by

$$\mathbb{E}_T(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + \frac{\delta}{2}|\nabla u|^2\right) dxdy\,, \tag{7}$$

where $\delta > 0$ is a parameter measuring the stretching capability of the plate. Then, if f denotes an external vertical load acting on the plate Ω and if u is the corresponding (small) deflection of the plate in the vertical direction, by (4) the total energy $\mathbb{E}_f$ of the plate becomes

$$\mathbb{E}_f(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + \frac{\delta}{2}|\nabla u|^2 \right) dxdy - \int_\Omega fu\, dxdy. \tag{8}$$

The unique minimizer u of $\mathbb{E}_f$ satisfies the Euler-Lagrange equation $\Delta^2 u - \delta\Delta u = f$ in Ω. We now determine the boundary conditions for a plate modeling a bridge. To get started, in Figure 1 we consider a (one dimensional) beam.

Figure 1: The depicted boundary condition for the left endpoints of these two cross sections is *clamped*, while the boundary conditions for the right endpoints are respectively *hinged* and *free*.

The function describing a clamped beam vanishes together with its first derivative at the endpoints of the beam. The function describing a hinged beam vanishes together with its second derivative at the endpoints of the beam. A free endpoint is obtained when no constraints are imposed.

A two dimensional rectangular plate has several resemblances to a beam due to the absence of curvatures on the boundary. The edges $x = 0$ and $x = \pi$ are hinged, since they are fixed to the ground and are free to rotate around the fixed edges in order to avoid bending of the plate close to these edges. Hence, we have

$$u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 \qquad \forall y \in (-\ell,\ell)\,. \tag{9}$$

No physical constraints are available on the edges $y = \pm\ell$ and the derivation of the boundary conditions there is much more involved, see [25, Section 6.48] (and also, more recently, [24, (2.40)]) for the case with no stretching $\delta = 0$. Here, we derive them while minimizing the energy functional $\mathbb{E}_f$, defined in (6), on the space

$$H^2_*(\Omega) := \Big\{ w \in H^2(\Omega);\ w = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell) \Big\}\,. \tag{10}$$

We also introduce

$$\mathcal{H} := \text{ the dual space of } H^2_*(\Omega)$$

and we denote by $\langle \cdot,\cdot \rangle$ the corresponding duality. Since we are in the plane, $H^2_*(\Omega) \subset C^0(\overline{\Omega})$ so that the condition on $\{0,\pi\} \times (-\ell,\ell)$ is satisfied pointwise. If $f \in L^1(\Omega)$, then the functional $\mathbb{E}_f$ is well-defined in $H^2_*(\Omega)$, otherwise we need to replace $\int fu$ with $\langle f,u \rangle$.

The energy (8) has been obtained after the linearization which led to approximate the stretching energy (2) with the Dirichlet integral in (3); this energy and the corresponding linear Euler-Lagrange equation are analyzed in the next section. If one avoids the approximation and considers the stretching energy (2), then the (SIM) is adopted and a quasilinear equation is obtained; this case will be analyzed in Section 4. Finally, in Section 5 we analyze the (NLM) which appears as an intermediate nonlinear case between the stretching energy (2) and the Dirichlet integral (3).

3 Analysis of the linearized model

The first somehow standard statement is the connection between minimizers of the energy function $\mathbb{E}_f$ and a suitable Euler-Lagrange equation.

Theorem 1. *Assume* (5) *and let $f \in \mathcal{H}$. Then there exists a unique $u \in H^2_*(\Omega)$ such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) + \delta \nabla u \cdot \nabla v]\, dxdy = \langle f, v\rangle \qquad \forall v \in H^2_*(\Omega)\,; \quad (11)$$

moreover, u is the minimum point of the convex functional $\mathbb{E}_f$. If $f \in L^2(\Omega)$ then $u \in H^4(\Omega)$. Finally, if $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfies (11)*, then u is a classical solution of*

$$\begin{cases} \Delta^2 u - \delta \Delta u = f & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \forall y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) - \delta u_y(x,\pm\ell) = 0 & \forall x \in (0,\pi). \end{cases} \quad (12)$$

As a consequence of Theorem 1, we see that the boundary conditions on the free edges read

$$u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0\,, \quad u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) - \delta u_y(x,\pm\ell) = 0 \qquad \forall x \in (0,\pi)\,. \quad (13)$$

Since we have in mind a long narrow rectangle, that is $\ell \ll \pi$, it is reasonable to assume that the forcing term will not depend on y. So, we now assume that

$$f = f(x)\,, \qquad f \in L^2(0,\pi). \quad (14)$$

In this case, we may solve (12) by separating variables. We first extend the source f as an odd 2π-periodic function over $\mathbb{R}$ and then expand it in a Fourier series

$$f(x) = \sum_{m=1}^{\infty} \beta_m \sin(mx)\,, \qquad \beta_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx\,, \quad (15)$$

where the series converges in $L^2(0,\pi)$ to f. Then we prove

Theorem 2. *Assume* (5) *and suppose that f satisfies* (14)-(15)*. Then the unique solution of* (12) *is given by*

$$u(x,y) = \sum_{m=1}^{\infty} \left[\frac{\beta_m}{m^2(m^2+\delta)} + A_m \cosh(my) + B_m \cosh(\sqrt{m^2+\delta}\,y)\right] \sin(mx)$$

where the coefficients $A_m = A_m(\ell)$ and $B_m = B_m(\ell)$ are defined by

$$\begin{aligned} A_m &= \frac{-\sigma(1-\sigma)m}{\sqrt{m^2+\delta}[(1-\sigma)m^2+\delta]^2 \sinh(m\ell)\coth(\ell\sqrt{m^2+\delta}) - (1-\sigma)^2(m^2+\delta)m^3\cosh(m\ell)}\,\beta_m \\ B_m &= \frac{\sigma[(1-\sigma)m^2+\delta]}{[(1-\sigma)m^2+\delta]^2\cosh(\ell\sqrt{m^2+\delta}) - (1-\sigma)^2 m^3\sqrt{m^2+\delta}\coth(m\ell)\sinh(\ell\sqrt{m^2+\delta})}\,\frac{\beta_m}{m^2+\delta}\,. \end{aligned} \quad (16)$$

Theorem 2 describes the dependence on y of the equilibrium when the forcing term merely depends on x. The y-dependence is a measure of the tendency to cross bending, which was the main cause of the collapse of the Tacoma Narrows Bridge, see [23]. The coefficients of β_m in (16) are negative for A_m and positive for B_m, and tend to vanish as $m \to \infty$. The tendency to cross bending is well estimated by

$$u(x,\ell) - u(x,0) = \sum_{m=1}^{\infty} \Big[A_m(\cosh(m\ell)-1) + B_m(\cosh(\ell\sqrt{m^2+\delta})-1)\Big]\sin(mx)\,.$$

In order to study the nonlinear problem, we need to set up a suitable functional framework. Let $D^2 w$ denote the Hessian matrix of a function $w \in H^2(\Omega)$. Thanks to the intermediate derivatives theorem, see [1, Theorem 4.15], the space $H^2(\Omega)$ is a Hilbert space if endowed with the scalar product

$$(u,v)_{H^2(\Omega)} := \int_\Omega \big(D^2 u \cdot D^2 v + uv\big)\, dxdy \qquad \text{for all } u,v \in H^2(\Omega)\,.$$

It is proved in [8, Lemma 4.1] that on the closed subspace $H^2_*(\Omega)$ (see (10)) we may also define a different scalar product. More precisely, we have

Proposition 3. *Assume* (5). *On the space* $H^2_*(\Omega)$ *the two norms*

$$w \mapsto \|w\|_{H^2(\Omega)}\,, \quad w \mapsto \|w\|_{H^2_*(\Omega)} := \left[\int_\Omega \left[(\Delta w)^2 + 2(1-\sigma)(w_{xy}^2 - w_{xx}w_{yy})\right] dxdy\right]^{1/2}$$

are equivalent. Therefore, $H^2_*(\Omega)$ *is a Hilbert space when endowed with the scalar product*

$$(u,v)_{H^2_*(\Omega)} := \int_\Omega \left[\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\right] dxdy\,. \tag{17}$$

We also need the space

$$H^1_*(\Omega) := \Big\{ w \in H^1(\Omega);\; w = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell) \Big\}\,. \tag{18}$$

Since $H^1_*(\Omega) \not\subset C^0(\overline{\Omega})$ we need to define this space in a more rigorous way. Consider the space

$$C^\infty_*(\Omega) := \Big\{ w \in C^\infty(\overline{\Omega});\; \exists\, \varepsilon > 0\,, w(x,y) = 0 \text{ if } x \in [0,\varepsilon] \cup [\pi-\varepsilon,\pi] \Big\}$$

which is a normed vector space when endowed with the Dirichlet norm

$$\|u\|_{H^1_*(\Omega)} = \left[\int_\Omega |\nabla u|^2\, dxdy\right]^{1/2}\,. \tag{19}$$

Then we define $H^1_*(\Omega)$ as the completion of $C^\infty_*(\Omega)$ with respect to the norm $\|\cdot\|_{H^1_*(\Omega)}$; the scalar product in $H^1_*(\Omega)$ is defined by

$$(u,v)_{H^1_*(\Omega)} := \int_\Omega \nabla u \nabla v\, dxdy \qquad \forall (u,v) \in H^1_*(\Omega)\,. \tag{20}$$

It is quite standard to prove that the embedding $H^2_*(\Omega) \subset H^1_*(\Omega)$ is compact and that the optimal embedding constant is given by

$$\Lambda := \min_{w \in H^2_*(\Omega)} \frac{\|w\|^2_{H^2_*(\Omega)}}{\|w\|^2_{H^1_*(\Omega)}}\,. \tag{21}$$

This yields the Poincaré-type inequality

$$\Lambda \|w\|^2_{H^1_*(\Omega)} \le \|w\|^2_{H^2_*(\Omega)} \qquad \forall w \in H^2_*(\Omega)\,; \tag{22}$$

this inequality is strict unless w minimizes the ratio in (21), that is, w is a nontrivial solution of the eigenvalue problem

$$\begin{cases} \Delta^2 u + \Lambda \Delta u = 0 & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \forall y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) + \Lambda u_y(x,\pm\ell) = 0 & \forall x \in (0,\pi). \end{cases} \tag{23}$$

In 1910, von Kármán [26] described the large deflections and stresses produced in a thin elastic plate subject to compressive forces along its edge by means of a system of two fourth order elliptic quasilinear equations. An interesting phenomenon associated with this nonlinear model is the appearance of "buckling", namely the plate may deflect out of its plane when these forces reach a certain magnitude. The linearization of the von Kármán equations for an elastic plate over a planar domain Ω under pressure also leads to the eigenvalue equation (23), although with different boundary conditions. Miersemann [20] studied in some detail the corresponding eigenvalue problem. We can also refer to [11, Section 1.3.2] for further details and more recent bibliography.

Note also that (23) is similar to (12) with $\delta = -\Lambda$ and $f = 0$. By strict monotonicity of the function (of variable λ) on the left hand side, we know that there exists a unique $\lambda \in (0, \sqrt{\sigma})$ such that

$$\frac{\lambda}{(\lambda^2 - \sigma)^2}\ \tanh(\ell\lambda) = \frac{1}{(1-\sigma)^2}\ \tanh(\ell)\,. \tag{24}$$

By taking advantage of Theorem 1, we obtain

Theorem 4. *Let* $\Lambda = \Lambda_1$ *be as in* (21) *and let* $\lambda \in (0, \sqrt{\sigma})$ *be the unique solution of* (24)*, then*

$$\Lambda = 1 - \lambda^2 \in (1-\sigma, 1)$$

and, up to a multiplicative constant, the unique solution of (23) *is positive in* Ω *and is given by*

$$\overline{w}(x,y) = \Big\{[\Lambda - (1-\sigma)]\ \cosh(\ell\sqrt{1-\Lambda})\ \cosh(y) + (1-\sigma)\ \cosh(\ell)\ \cosh(y\sqrt{1-\Lambda})\Big\}\sin x\,. \tag{25}$$

Moreover, (23) *admits a sequence of divergent eigenvalues*

$$\Lambda_1 < \Lambda_2 \leq ... \leq \Lambda_k \leq ... \tag{26}$$

whose corresponding eigenfunctions $\{\overline{w}_k\}$ *form a complete orthonormal system in* $H^2_*(\Omega)$.

Theorem 4 is by far nontrivial. First of all, the computation of the exact value of the least eigenvalue requires some effort; if $\sigma = 1/3$ and $\ell = \pi/200$ then $\Lambda \approx 0.889$. Secondly, and more important, it is well-known that the first eigenfunction of some biharmonic problems may change sign; when Ω is a square, Coffman [7] proved that the first eigenfunction of the clamped plate problem changes sign, see also [14] for more general results and [11, Section 3.1.3] for the updated history of the problem. Hence, the positivity of $\overline{w}$ was not expected. Last but not least, note that (23) is not a standard eigenvalue problem such as $Lu = \Lambda u$ for some linear operator L; some work is needed in order to exhibit a linear compact and self-adjoint operator.

4 The surface increment quasilinear model (SIM)

We study here the behavior of the plate without the approximation (3):

$$\begin{aligned}\mathcal{E}_f(u) &= \int_\Omega \left(\frac{|\Delta u|^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - P\left(\sqrt{1+|\nabla u|^2} - 1\right)\right) dxdy - \langle f, u\rangle \\ &= \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{P}{2}\|u\|^2_{H^1_*(\Omega)} + P\int_\Omega N(|\nabla u|)dxdy - \langle f, u\rangle\end{aligned} \tag{27}$$

where $\|\cdot\|_{H^2_*(\Omega)}$ is the norm defined in Proposition 3, $\|\cdot\|_{H^1_*(\Omega)}$ is the Dirichlet norm defined in (19), and

$$N(s) = \frac{1}{2}s^2 - \sqrt{1+s^2} + 1 = \frac{1}{8}s^4 - \frac{1}{16}s^6 + O\left(s^8\right). \tag{28}$$

Notice that,

$$N(s) \geq 0 \quad \text{and} \quad N''(s) \geq 0\,. \tag{29}$$

The constant $P > 0$ is the axial force acting on the short edges of the plate (prestressing). We have $P > 0$ if the plate is compressed and $P < 0$ if the plate is stretched. The energy function $\mathcal{E}_f$ may not be convex, hence it may admit critical points different from the global minimizer: although the only stable equilibrium is the global minimizer, other unstable equilibria may exist. By arguing as for Theorem 1 one can verify that the critical points of the energy $\mathcal{E}_f(u)$ solve the following (nonlinear) Euler-Lagrange equation:

$$\begin{cases}\Delta^2 u + P\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f & \text{in } \Omega\\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \forall y \in (-\ell,\ell)\\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) - \delta(u)u_y(x,\pm\ell) = 0 & \forall x \in (0,\pi)\end{cases} \tag{30}$$

where

$$\delta(u) = -\frac{P}{\sqrt{1+|\nabla u|^2}}\,.$$

For a given $f \in \mathcal{H}$ we say that $u \in H^2_*(\Omega)$ is a weak solution of (30) if

$$(u,v)_{H^2_*(\Omega)} - P\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}dxdy = \langle f, v\rangle \qquad \forall v \in H^2_*(\Omega)\,. \tag{31}$$

First we consider the homogeneous case $f = 0$. In this case, we say that $u \in H^2_*(\Omega)$ is a weak solution of (30) if

$$(u,v)_{H^2_*(\Omega)} - P\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}dxdy = 0 \qquad \forall v \in H^2_*(\Omega)\,. \tag{32}$$

Of course, $u = 0$ always solves (32) and the interesting question is whether nontrivial solutions also exist. In this respect, we prove

Theorem 5. *Let Λ_k ($k \geq 1$) be as in* (26) *and put $\Lambda_0 = 0$. If*

$$P \in (\Lambda_k, \Lambda_{k+1}] \tag{33}$$

for some $k \geq 0$, then (32) *admits at least k pairs of nontrivial solutions $\pm u_j$.*

The picture in Figure 2 displays the qualitative behavior of the energy functional $\mathcal{E}_0$ when $P \in (\Lambda_1, \Lambda_2]$. There is an unstable equilibrium (at the origin) and two stable equilibria symmetric with respect to the origin.

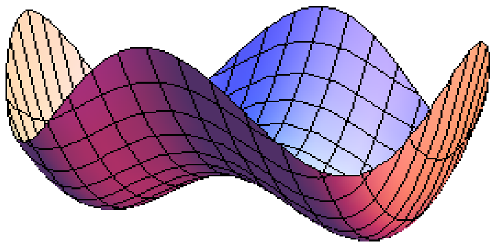

Figure 2: Qualitative description of the energy $\mathcal{E}_0$ with three equilibria.

For $P > \Lambda_2$, Theorem 5 gives additional multiplicity and the qualitative graph of $\mathcal{E}_0$ becomes more complicated, with more unstable solutions.

For the nonhomogeneous problem our result is less precise. However, also when $f \neq 0$ we may state a result about uniqueness and multiplicity of solutions.

Theorem 6. *Let Λ be as in* (21) *(see Theorem 4).*

(i) If $P \leq \Lambda$, then for all $f \in \mathcal{H}$ the problem (32) *admits a unique solution $u \in H^2_*(\Omega)$;*

(ii) If $P > \Lambda$, then for all $f \in \mathcal{H}$ the problem (32) *admits at least a solution $u \in H^2_*(\Omega)$;*

(iii) If $P > \Lambda$, there exists $\gamma = \gamma(P) > 0$ such that if $\|f\|_{\mathcal{H}} < \gamma$ then (32) *admits at least* 3 *solutions.*

Theorem 6 deserves several important comments.

From a physical point of view, it is not unexpected. If the prestressing constant is sufficiently small (that is, $P \leq \Lambda$), then there exists a unique equilibrium position for any given external forcing and this equilibrium is stable. But if prestressing is sufficiently large (that is, $P > \Lambda$) then other equilibria may appear. We conjecture that for increasing P (and given small f) the number of solutions also increases.

For the proof of Theorem 6 we use a perturbation argument. Therefore, if $P \in (\Lambda_1, \Lambda_2]$ the picture in Figure 2 still displays the qualitative behavior of the energy functional $\mathcal{E}_f$: there is an unstable equilibrium (close to the origin) and two stable equilibria (nearly symmetric with respect to the origin). For larger P, further solutions may appear but they cannot be directly derived from Theorem 5 due to the instability of the solutions found there. Of course, if f belongs to some space related to the kernel of the second derivative of $\mathcal{E}_0$ one may also maintain the unstable solutions of the homogeneous problem; but we will not pursue this further.

The classical Fredholm alternative tells us that nonhomogeneous linear problems at resonance lack either existence or uniqueness of the solution. The simplest relevant example is the equation $\Delta u + \lambda u = f$ in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) under Dirichlet boundary conditions; if λ is an eigenvalue of the Laplacian, that is $-\Delta u = \lambda u$ admits a nontrivial solution $\overline{w} \in H^1_0(\Omega)$, the existence of solutions of the nonhomogeneous problem depends on $f \in H^{-1}(\Omega)$: there are no solutions if $\langle f, \overline{w}\rangle \neq 0$ and there are infinitely many solutions otherwise. Theorem 6 (and Theorem 5) tells us that (30) admits a unique solution also at resonance, when $P = \Lambda$, for *any* $f \in \mathcal{H}$; this confirms the nonlinear nature of (30).

5 The nonlocal quasilinear model (NLM)

We study here the behavior of the plate subject to prestressing and we follow the model suggested by Berger [3]: see also the previous model for a beam suggested by Woinowsky-Krieger [27]. The elastic energy to be considered reads

$$\begin{aligned}\mathcal{E}_f(u) &= \int_\Omega \left(\frac{|\Delta u|^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - \frac{P}{2}|\nabla u|^2\right) dxdy + \frac{S}{4}\left(\int_\Omega |\nabla u|^2 dxdy\right)^2 - \langle f, u\rangle \\ &= \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{P}{2}\|u\|^2_{H^1_*(\Omega)} + \frac{S}{4}\|u\|^4_{H^1_*(\Omega)} - \langle f, u\rangle \end{aligned} \tag{34}$$

where $\|\cdot\|_{H^2_*(\Omega)}$ is the norm defined in Proposition 3 and $\|\cdot\|_{H^1_*(\Omega)}$ is the Dirichlet norm defined in (19). Here $S > 0$ depends on the elasticity of the material composing the roadway, $S\int_\Omega |\nabla u|^2$ measures the geometric nonlinearity of the plate due to its stretching, and $P > 0$ is the prestressing constant. Again, we have $P > 0$ if the plate is compressed and $P < 0$ if the plate is stretched.

By arguing as in the proof of Theorem 1, one sees that the critical points of the energy $\mathcal{E}_f(u)$ solve the following (nonlinear, nonlocal) Euler-Lagrange equation:

$$\Delta^2 u + \Big(P - S\int_\Omega |\nabla u|^2 dxdy\Big)\Delta u = f \quad \text{in } \Omega\,. \tag{35}$$

Note that the second order expansion of $N(s)$ in (28) leads to an energy similar to (34) with

$$\frac{P}{8}\int_\Omega |\nabla u|^4 dxdy \quad \text{instead of} \quad \frac{S}{4}\left(\int_\Omega |\nabla u|^2 dxdy\right)^2 ; \tag{36}$$

these terms are quite similar and hence the Berger energy (34) may be considered as a second order (nonlinear) approximation of (27). Using (36) yields the quasilinear equation

$$\Delta^2 u + P\Delta u - \frac{P}{2}\Delta_4 u = f \quad \text{in } \Omega$$

where Δ_4 is the p-Laplacian operator with $p = 4$; this equation should be compared with (35).

For simplicity we put

$$\delta(u) = -P + S\int_\Omega |\nabla u|^2\, dxdy \tag{37}$$

so that the corresponding boundary value problem for (35) reads

$$\begin{cases} \Delta^2 u - \delta(u)\,\Delta u = f & \text{in } \Omega \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \forall y \in (-\ell,\ell) \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) - \delta(u)u_y(x,\pm\ell) = 0 & \forall x \in (0,\pi)\,. \end{cases} \tag{38}$$

For a given $f \in \mathcal{H}$ we say that $u \in H^2_*(\Omega)$ is a weak solution of (38) if

$$(u,v)_{H^2_*(\Omega)} + \delta(u)(u,v)_{H^1_*(\Omega)} = \langle f, v\rangle \qquad \forall v \in H^2_*(\Omega)\,. \tag{39}$$

It appears quite instructive to deal first with the homogeneous case $f = 0$: the energy functional $\mathcal{E}_0$ has a rich structure. In this case, we say that $u \in H^2_*(\Omega)$ is a weak solution of (38) if

$$(u,v)_{H^2_*(\Omega)} + \delta(u)(u,v)_{H^1_*(\Omega)} = 0 \qquad \forall v \in H^2_*(\Omega)\,. \tag{40}$$

We then prove a precise multiplicity result.

Theorem 7. *Let $S > 0$, let Λ_k ($k \geq 1$) be as in* (26) *and put $\Lambda_0 = 0$.*

If $P \in (\Lambda_k, \Lambda_{k+1}]$ for some $k \geq 0$ and at least one of the eigenvalues $\Lambda_2, ..., \Lambda_k$ has multiplicity larger than 1*, then* (40) *admits infinitely many solutions.*

If $P \in (\Lambda_k, \Lambda_{k+1}]$ for some $k \geq 0$ and all the eigenvalues $\Lambda_2, ..., \Lambda_k$ have multiplicity 1*, then* (40) *admits exactly $2k + 1$ solutions which are explicitly given by*

$$u_0 = 0\,, \qquad \pm u_j = \pm\sqrt{\frac{P-\Lambda_j}{S}}\,\overline{w}_j \quad (j = 1, ..., k) \tag{41}$$

where $\overline{w}_j$ is an $H^1_*(\Omega)$-normalized eigenfunction of (23) *corresponding to the eigenvalue Λ_j ($j = 1, ..., k$). Moreover, for each solution the energy is given by*

$$\mathcal{E}_0(u_0) = 0\,, \qquad \mathcal{E}_0(\pm u_j) = -\frac{(P-\Lambda_j)^2}{4S} \quad (j = 1, ..., k) \tag{42}$$

and the Morse index M is given by

$$M(u_0) = k\,, \qquad M(\pm u_j) = j - 1 \quad (j = 1, ..., k)\,. \tag{43}$$

The statement of Theorem 7 can be made more precise in the case of infinitely many solutions. Any simple eigenvalue Λ_j generates solutions satisfying (42) and (43). Multiple eigenvalues Λ_j generate infinitely many solutions which are linear combinations of the (multiple) related eigenfunctions, they still satisfy (42) but they are degenerate critical points for $\mathcal{E}_0$.

The picture in Figure 2 displays again the qualitative behavior of the energy functional $\mathcal{E}_0$ when $P \in (\Lambda_1, \Lambda_2]$. However, in this case more details are available. If Λ_2 is simple, the picture of Figure 3 represents the 2D space

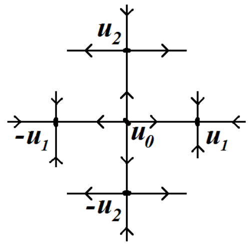

Figure 3: Qualitative description of the energy $\mathcal{E}_0$ with five equilibria.

spanned by the first two eigenfunctions $\{\overline{w}_1, \overline{w}_2\}$ and the five solutions $\{u_0, \pm u_1, \pm u_2\}$; the arrows represent the directions where the energy $\mathcal{E}_0$ decreases. Theorem 7 (in particular, (43)) states that only $\pm u_1$ are stable (global minima) while all the other critical points of $\mathcal{E}_0$ are saddle points. If $P \in [\Lambda_2, \Lambda_3]$, in the orthogonal complementary space of codimension 2, the trivial solution u_0 is a global minimum for $\mathcal{E}_0$. By standard minimax techniques, one could try to link the two minima $\pm u_1$ in order to find further unstable (mountain pass) solutions. But Theorem 7 states that the only solutions are (41) so that the minimax point is one of the u_j's ($j = 0, ..., k$) and no additional unstable solutions exist.

For the nonhomogeneous problem the result is similar to Theorem 6:

Theorem 8. *Let $S > 0$ and let Λ be as in* (21) *(see Theorem 4).*

(i) *If $P \le \Lambda$, then for all $f \in \mathcal{H}$ the problem* (38) *admits a unique weak solution $u \in H^2_*(\Omega)$.*

(ii) *If $P > \Lambda$, then for all $f \in \mathcal{H}$ the problem* (38) *admits at least a weak solution $u \in H^2_*(\Omega)$; moreover, there exists $\gamma = \gamma(P, k) > 0$ such that if $\|f\|_{\mathcal{H}} < \gamma$ then* (38) *admits at least* 3 *weak solutions.*

The same comments and remarks following Theorem 6 apply also to Theorem 8. But a further remark about a priori estimates may be given here. Of course, this makes sense in the case of a unique solution $P \le \Lambda$. By taking $v = u$ in (39), we obtain

$$\|u\|^2_{H^2_*(\Omega)} - P\|u\|^2_{H^1_*(\Omega)} + S\|u\|^4_{H^1_*(\Omega)} = \langle f, u\rangle \le \|f\|_{\mathcal{H}}\|u\|_{H^2_*(\Omega)}\,. \tag{44}$$

If $P < \Lambda$, then by (22) this yields the following a priori estimate for the unique solution of (38):

$$\|u\|_{H^2_*(\Omega)} \le \frac{\Lambda}{\Lambda - P}\|f\|_{\mathcal{H}}\,. \tag{45}$$

As $P \to \Lambda$ this estimate is of little interest and one may wonder whether a different a priori estimate may be obtained for $P = \Lambda$. But this is not the case, as shown by the following counterexample. Let $\overline{w}$ denote an $H^1_*(\Omega)$-normalized eigenfunction corresponding to Λ and let $f = \alpha\Delta\overline{w}$ for some $\alpha > 0$. By Theorem 8, the problem (38) admits a unique solution; one may check that

$$u = -\sqrt[3]{\frac{\alpha}{S}}\,\overline{w}$$

is the solution. Then

$$\|u\|_{H^2_*(\Omega)} = \sqrt{\Lambda}\,\|u\|_{H^1_*(\Omega)} = \frac{\sqrt{\Lambda}}{\sqrt[3]{S}}\,\sqrt[3]{\alpha}\,, \qquad \|f\|_{\mathcal{H}} = \alpha\|\Delta\overline{w}\|_{\mathcal{H}}\,,$$

so that no a priori estimate such as $\|u\|_{H^2_*(\Omega)} \le C\|f\|_{\mathcal{H}}$ can hold; to be convinced, it suffices to let $\alpha \to 0$. On the other hand, for any $P > 0$, from (44) we infer that

$$\text{either} \qquad \|u\|_{H^1_*(\Omega)} \le \sqrt{\frac{P}{S}} \qquad \text{or} \qquad \|u\|_{H^2_*(\Omega)} \le \|f\|_{\mathcal{H}}$$

so that, by (21),

$$\|u\|_{H^1_*(\Omega)} \le \max\left\{\sqrt{\frac{P}{S}}, \frac{\|f\|_{\mathcal{H}}}{\sqrt{\Lambda}}\right\}$$

and a kind of priori estimate in the weaker norm $H^1_*(\Omega)$ is always available.

Note that by taking $v = u$ in (31), if $P < \Lambda$, we also obtain the a priori estimate (45) for the unique solution of (30). But contrary to (38), nothing can be said when $P \to \Lambda$ or when $P = \Lambda$.

6 Proof of Theorem 1

By Proposition 3, the bilinear form (17) is a scalar product over $H^2_*(\Omega)$. Therefore, since $\delta > 0$, the bilinear form defined in the right hand side of (11) is coercive (and symmetric). By the Lax-Milgram Theorem, we infer that for any $f \in \mathcal{H}$ there exists a unique $u \in H^2_*(\Omega)$ satisfying (11). This proves the first part of Theorem 1.

We now show that smooth weak solutions and classical solutions coincide and we justify the boundary conditions. If $u \in H^2_*(\Omega)$, then u vanishes identically on the two short edges of Ω and hence

$$u(0, y) = u(\pi, y) = u_y(0, y) = u_y(\pi, y) = u_{yy}(0, y) = u_{yy}(\pi, y) = 0 \qquad \text{for } y \in (-\ell, \ell)\,. \tag{46}$$

Recalling the Gauss-Green formula

$$\int_\Omega \Delta u \Delta v\, dx\, dy = \int_\Omega \Delta^2 uv\, dx\, dy + \int_{\partial\Omega} \Big[\Delta u\, v_\nu - v\,(\Delta u)_\nu\Big]\, ds$$

and with some integration by parts, we obtain that if $u \in C^4(\overline{\Omega}) \cap H^2_*(\Omega)$ satisfies (11), then

$$\int_\Omega (\Delta^2 u - \delta\Delta u - f)v\, dxdy + \int_{-\ell}^{\ell} [u_{xx}(\pi,y)v_x(\pi,y) - u_{xx}(0,y)v_x(0,y)]\, dy \tag{47}$$
$$+\int_0^\pi \Big\{ [u_{yyy}(x,-\ell) + (2-\sigma)u_{xxy}(x,-\ell) - \delta u_y(x,-\ell)]\, v(x,-\ell) - [u_{yy}(x,-\ell) + \sigma u_{xx}(x,-\ell)]\, v_y(x,-\ell)\Big\} dx$$
$$+\int_0^\pi \Big\{ [u_{yy}(x,\ell) + \sigma u_{xx}(x,\ell)]\, v_y(x,\ell) - [u_{yyy}(x,\ell) + (2-\sigma)u_{xxy}(x,\ell) - \delta u_y(x,\ell)]\, v(x,\ell)\Big\} dx = 0$$

for any $v \in H^2_*(\Omega)$. If we choose $v \in C^2_c(\Omega)$ in (47), then all the boundary terms vanish and we deduce that $\Delta^2 u = f$ in Ω. Hence we may drop the double integral in (47). By the arbitrariness of v, the coefficients of the terms $v_x(\pi,y)$, $v_x(0,y)$, $v(x,-\ell)$, $v_y(x,-\ell)$, $v_y(x,\ell)$, and $v(x,\ell)$ must vanish identically and we obtain (9)-(13). We have so proved that if $u \in C^4(\overline{\Omega}) \cap H^2_*(\Omega)$ satisfies (11) then it is a classical solution of (12).

7 Proof of Theorem 2

We make use of separation of variables methods. See [8, 16, 18, 21, 28] for the same technique applied to simpler problems for rectangular plates. We introduce the auxiliary function

$$\phi(x) := \sum_{m=1}^\infty \frac{\beta_m}{m^2(m^2+\delta)} \sin(mx) \tag{48}$$

and we observe that it solves the following boundary value problem for an ordinary differential equation

$$\phi''''(x) - \delta\phi''(x) = f(x) \quad \text{in } (0,\pi)\,, \qquad \phi(0) = \phi''(0) = \phi(\pi) = \phi''(\pi) = 0\,.$$

Moreover, $\phi''^2(0,\pi)$ is given by

$$\phi''(x) = -\sum_{m=1}^\infty \frac{\beta_m}{m^2+\delta} \sin(mx) \tag{49}$$

and the series (49) converges in $H^2(0,\pi)$ and, hence, uniformly.

We now define $v(x,y) := u(x,y) - \phi(x)$ so that if u solves (12), then v satisfies

$$\begin{cases} \Delta^2 v - \delta\Delta v = 0 & \text{in } \Omega \\ v = v_{xx} = 0 & \text{on } \{0,\pi\}\times(-\ell,\ell) \\ v_{yy} + \sigma v_{xx} = -\sigma\phi'' & \text{on } (0,\pi)\times\{-\ell,\ell\} \\ v_{yyy} + (2-\sigma)v_{xxy} - \delta v_y = 0 & \text{on } (0,\pi)\times\{-\ell,\ell\}\,. \end{cases} \tag{50}$$

We seek solutions of (50) by separating variables, namely we seek functions $V_m = V_m(y)$ such that

$$v(x,y) = \sum_{m=1}^\infty V_m(y)\sin(mx)$$

solves (50). Then

$$\Delta^2 v(x,y) - \delta\Delta v(x,y) = \sum_{m=1}^\infty [V_m''''(y) - (2m^2+\delta)V_m''(y) + (m^4+\delta m^2)V_m(y)]\sin(mx)$$

and the equation in (50) yields

$$V_m''''(y) - (2m^2 + \delta)V_m''(y) + (m^4 + \delta m^2)V_m(y) = 0 \quad \text{for } y \in (-\ell, \ell) \, . \tag{51}$$

The characteristic equation of (51) is $\lambda^4 - (2m^2+\delta)\lambda^2 + m^4 + \delta m^2 = 0$ and therefore $\lambda \in \{\pm m, \pm\sqrt{m^2+\delta}\}$. Whence, the solutions of (51) are linear combinations of

$$\{\cosh(my), \sinh(my), \cosh(\sqrt{m^2+\delta}\, y), \sinh(\sqrt{m^2+\delta}\, y)\}$$

but, due to the symmetry of Ω and to the uniqueness of the solution v of (50), we know that V_m is even with respect to y. Hence, we seek functions V_m of the form

$$V_m(y) = A_m \cosh(my) + B_m \cosh(\sqrt{m^2+\delta}\, y) \tag{52}$$

where $A_m = A_m(\ell)$ and $B_m = B_m(\ell)$ have to be determined by imposing the boundary conditions in (50). This yields

$$\begin{aligned} \big[m^2(1-\sigma)\big] A_m \cosh m\ell + \big[m^2(1-\sigma)+\delta\big] B_m \cosh \widetilde{m}\ell &= \sigma\beta_m/\widetilde{m}^2 \\ m\big[m^2(1-\sigma)+\delta\big] A_m \sinh m\ell + \widetilde{m} m^2 (1-\sigma) B_m \sinh \widetilde{m}\ell &= 0 \end{aligned}$$

where $\widetilde{m} = \sqrt{m^2+\delta}$. To simplify further the notation, set $a_m = m^2(1-\sigma)$ and $b_m = \sigma\beta_m/\widetilde{m}^2$ in this pair of equations, so that

$$\begin{aligned} a_m \cosh m\ell \, A_m + (a_m+\delta)\cosh \widetilde{m}\ell \, B_m &= b_m \\ m(a_m+\delta)\sinh m\ell \, A_m + \widetilde{m} a_m \sinh \widetilde{m}\ell \, B_m &= 0. \end{aligned}$$

The determinant of this system is

$$C_m = \widetilde{m} a_m^2 \cosh m\ell \sinh \widetilde{m}\ell - m(a_m+\delta)^2 \sinh m\ell \cosh \widetilde{m}\ell.$$

It can be shown (for instance, by using (54) as in the proof of Lemma 9 below) that $C_m < 0$ for all positive integers m. Therefore

$$A_m = \frac{\widetilde{m} a_m \sinh \widetilde{m}\ell}{C_m} b_m \, , \quad B_m = \frac{-m(a_m+\delta)\sinh m\ell}{C_m} b_m$$

and (16) follows.

8 Proof of Theorem 4

We first note that $\eta(x,y) := \sin x$ satisfies

$$\eta \in H^2_*(\Omega) \, , \qquad \frac{\|\eta\|^2_{H^2_*(\Omega)}}{\|\eta\|^2_{H^1_*(\Omega)}} = 1 \, .$$

Moreover, η does not satisfy the boundary conditions in (23) so that it is not an eigenfunction. Combined with (21), these facts show that

$$\Lambda < 1 \, . \tag{53}$$

Next, we note that a simple differentiation yields

$$\frac{d}{dt}\Big(\tanh(t) - t(1-\tanh^2(t)) \Big) = 2t \tanh(t)(1-\tanh^2(t)) > 0$$

and therefore

$$t(1-\tanh^2(t)) < \tanh(t) \qquad \forall t>0\,. \tag{54}$$

We seek solutions of (23) by separating variables, that is, in the form

$$u(x,y)=\sum_{m=1}^{\infty} V_m(y)\sin(mx)\,.$$

Then

$$\Delta^2 u+\Lambda\Delta u=\sum_{m=1}^{\infty}[V_m''''(y)+(\Lambda-2m^2)V_m''(y)+m^2(m^2-\Lambda)V_m(y)]\sin(mx)$$

and we are led to find nontrivial solutions $V_m\in C^\infty(-\ell,\ell)$ of the equation

$$V_m''''(y)+(\Lambda-2m^2)V_m''(y)+m^2(m^2-\Lambda)V_m(y)=0 \qquad y\in(-\ell,\ell)\,. \tag{55}$$

We derive the boundary conditions for V_m from (23) and we obtain

$$V_m''(\pm\ell)-\sigma m^2 V_m(\pm\ell)=0\,, \qquad V_m'''(\pm\ell)-\Big((2-\sigma)m^2-\Lambda\Big)V_m'(\pm\ell)=0\,. \tag{56}$$

Since they are evaluated at $\pm\ell$ these boundary conditions will enable us to separate between the even and odd terms.

The characteristic equation associated with (55) reads

$$\alpha^4+(\Lambda-2m^2)\alpha^2+m^2(m^2-\Lambda)=0 \implies (\alpha^2-m^2)(\alpha^2+\Lambda-m^2)=0\,. \tag{57}$$

The estimate (53) tells us that all the solutions to (57) are real numbers and are given by $\alpha\in\{\pm m,\pm\sqrt{m^2-\Lambda}\}$. Hence, the general solution of (55) reads

$$V_m(y)=a\,\cosh(my)+b\,\sinh(my)+c\,\cosh(y\sqrt{m^2-\Lambda})+d\,\sinh(y\sqrt{m^2-\Lambda}) \qquad (a,b,c,d\in\mathbb{R})\,.$$

By (53) we also know that

$$\exists\lambda\in(0,1)\,,\quad \Lambda=(1-\lambda^2)m^2$$

and the general solution of (55) becomes

$$V_m(y)=a\,\cosh(my)+b\,\sinh(my)+c\,\cosh(m\lambda y)+d\,\sinh(m\lambda y) \qquad (a,b,c,d\in\mathbb{R})\,, \tag{58}$$

so that

$$\begin{aligned} V_m''(y)-\sigma m^2V_m(y) &= m^2\Big[(1-\sigma)a\,\cosh(my)+(\lambda^2-\sigma)c\,\cosh(m\lambda y)\\ &\quad +(1-\sigma)b\,\sinh(my)+(\lambda^2-\sigma)d\,\sinh(m\lambda y)\Big]\,,\\ V_m'''(y)-\Big((2-\sigma)m^2-\Lambda\Big)V_m'(y) &= m^3\Big[(\sigma-\lambda^2)a\,\sinh(my)-\lambda(1-\sigma)c\,\sinh(m\lambda y)\\ &\quad +(\sigma-\lambda^2)b\,\cosh(my)-\lambda(1-\sigma)d\,\cosh(m\lambda y)\Big]\,. \end{aligned}$$

Hence, by imposing the boundary conditions (56) and by separating the odd and even terms, we get

$$\begin{cases} (1-\sigma)\cosh(m\ell)\,a+(\lambda^2-\sigma)\cosh(m\ell\lambda)\,c=0\\ (\lambda^2-\sigma)\sinh(m\ell)\,a+\lambda(1-\sigma)\sinh(m\ell\lambda)\,c=0\,, \end{cases} \tag{59}$$

$$\begin{cases} (1-\sigma)\sinh(m\ell)\,b+(\lambda^2-\sigma)\sinh(m\ell\lambda)\,d=0\\ (\lambda^2-\sigma)\cosh(m\ell)\,b+\lambda(1-\sigma)\cosh(m\ell\lambda)\,d=0\,. \end{cases} \tag{60}$$

We are seeking nontrivial functions V_m of the form (58) and hence nontrivial solutions a,b,c,d of the above systems. We first prove

Lemma 9. *For any $m \in \mathbb{N}$ there exists a unique $\lambda_m \in (0,1)$ such that (59) admits a nontrivial solution (a,c). Moreover, $\lambda_m \in (0,\sqrt{\sigma})$ and $(1-\lambda_m^2)m^2 > 1-\lambda_1^2$ for all $m \geq 2$.*

Proof. The determinant of (59) vanishes if and only if

$$f(\lambda) = f(1)\,, \tag{61}$$

where

$$f(s) := \frac{s}{(s^2-\sigma)^2}\tanh(m\ell s) \qquad \forall s \in [0,\sqrt{\sigma})\cup(\sqrt{\sigma},1]\,. \tag{62}$$

We claim that there exists a unique $\lambda \in (0,1)$ such that (61) holds. To see this, we note that

$$f'(s) = \frac{3s^2+\sigma}{(\sigma-s^2)^3}\tanh(m\ell s) + \frac{m\ell s}{(\sigma-s^2)^2}\Big[1-\tanh^2(m\ell s)\Big] \qquad \forall s \in (0,\sqrt{\sigma})\cup(\sqrt{\sigma},1)\,.$$

Therefore, $f'(s) > 0$ for all $s \in (0,\sqrt{\sigma})$ since f' is the sum of two positive terms. On the other hand, by (54) we have

$$f'(s) < -\frac{3s^2+\sigma}{(s^2-\sigma)^3}\tanh(m\ell s) + \frac{\tanh(m\ell s)}{(s^2-\sigma)^2} = -2\frac{s^2+\sigma}{(s^2-\sigma)^3}\tanh(m\ell s) < 0 \qquad \forall s \in (\sqrt{\sigma},1)\,.$$

Hence, f is strictly increasing in $(0,\sqrt{\sigma})$ and strictly decreasing in $(\sqrt{\sigma},1)$; its graph is qualitatively as in

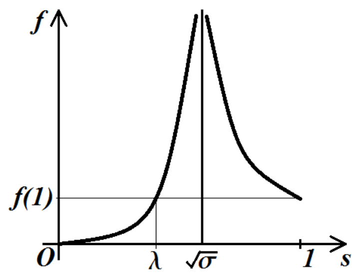

Figure 4: Qualitative plot of the graph of f, the function defined in (62).

Figure 4. This proves that for any $m \geq 1$ there exists a unique $\lambda_m \in (0,1)$ such that (61) holds. Moreover, $\lambda_m \in (0,\sqrt{\sigma})$. This completes the first part of the proof.

Next we remark that, by (5) and what we just proved, for any $m \geq 2$ we have $(1-\lambda_m^2)m^2 > 4(1-\sigma) > 2$. This shows that $(1-\lambda_m^2)m^2 > 1-\lambda_1^2$ and completes the proof. □

Now we prove

Lemma 10. *For any $m \in \mathbb{N}$ there exists at most one $\lambda^m \in (0,1)$ such that (60) admits a nontrivial solution (b,d). Moreover, if such λ^m exists and if λ_1 is as in Lemma 9 then $(1-(\lambda^m)^2)m^2 > 1-\lambda_1^2$ for all $m \geq 1$.*

Proof. The determinant of (60) vanishes if and only if

$$g(\lambda) = g(1) \tag{63}$$

where

$$g(s) = \frac{(s^2-\sigma)^2}{s}\tanh(m\ell s) \qquad \forall s \in (0,1]\,. \tag{64}$$

By differentiating we find

$$g'(s) = \frac{(s^2-\sigma)(3s^2+\sigma)}{s^2}\tanh(m\ell s) + \frac{(s^2-\sigma)^2}{s}\,m\ell\Big[1-\tanh^2(m\ell s)\Big] \qquad \forall s \in (0,1]\,.$$

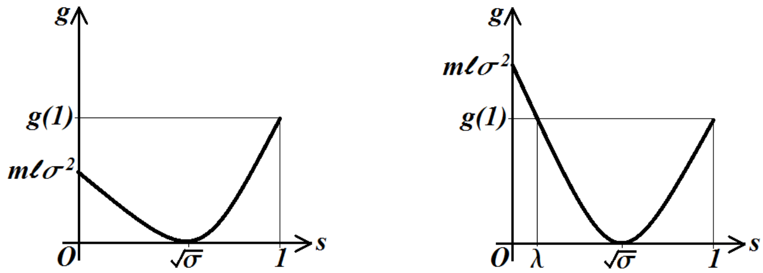

Figure 5: Qualitative plot of the graph of g, the function defined in (64).

If $s > \sqrt{\sigma}$ then $g'(s) > 0$ since it is the sum of two positive terms. If $s < \sqrt{\sigma}$ then by (54) we get

$$g'(s) < 4(s^2 - \sigma)\tanh(m\ell s) < 0 \qquad \forall s \in (0, \sqrt{\sigma})\,.$$

Therefore, the graph of g is qualitatively as in Figure 5. If $\lim_{s\to 0} g(s) = m\ell\sigma^2 \le (1-\sigma)^2\tanh(m\ell) = g(1)$ then no $\lambda \in (0,1)$ satisfying (63) exist (see the left picture in Figure 5). If the opposite inequality holds (a case which certainly occurs for large m) then there exists a unique $\lambda^m \in (0,1)$ satisfying (63) and, moreover, $\lambda^m \in (0, \sqrt{\sigma})$ (see the right picture in Figure 5). In particular, as in the proof of Lemma 10, this tells us that $(1-(\lambda^m)^2)m^2 > 1 - \lambda_1^2$ for all $m \ge 2$ for which λ^m exists. So, it remains to prove that $1-(\lambda^1)^2 > 1-\lambda_1^2$ if λ^1 exists. Recalling that λ^1 satisfies (63) for $m = 1$, we have

$$\begin{aligned} \frac{((\lambda^1)^2-\sigma)^2}{\lambda^1} &= (1-\sigma)^2\,\frac{\tanh(\ell)}{\tanh(\ell\lambda^1)} \\ \text{by (61)} \quad &= \frac{\tanh^2(\ell)}{\tanh(\ell\lambda^1)\,\tanh(\ell\lambda_1)}\,\frac{(\lambda_1^2-\sigma)^2}{\lambda_1} \\ \text{by monotonicity of tanh} \quad &> \frac{(\lambda_1^2-\sigma)^2}{\lambda_1}\,. \end{aligned}$$

Since $s \mapsto \frac{(s^2-\sigma)^2}{s}$ is decreasing in $(0,\sqrt{\sigma})$ this shows that $\lambda^1 < \lambda_1$ and then that $1-(\lambda^1)^2 > 1-\lambda_1^2$. □

By Lemmas 9 and 10 we know that the least eigenvalue Λ for (23) is given by $\Lambda = 1-\lambda^2$ where $\lambda = \lambda_1$ is the unique solution of (61) when $m = 1$, that is,

$$\frac{\lambda}{(\lambda^2-\sigma)^2}\,\tanh(\ell\lambda) = \frac{1}{(1-\sigma)^2}\,\tanh(\ell)\,.$$

In this case, the determinant of (59) vanishes and a possible choice of nontrivial a, c leads to a nontrivial function V_1 of the form

$$V_1(y) = [\Lambda - (1-\sigma)]\,\cosh(\ell\sqrt{1-\Lambda})\,\cosh(y) + (1-\sigma)\,\cosh(\ell)\,\cosh(y\sqrt{1-\Lambda})\,.$$

By multiplying this function by $\sin x$ we obtain $\overline{w}$ as in (25).

Finally, note that with the scalar products defined in (17) and (20) the eigenvalue problem (23) reads

$$(u,v)_{H^1_*(\Omega)} = \frac{1}{\Lambda}(u,v)_{H^2_*(\Omega)} \qquad \forall v \in H^2_*(\Omega)\,.$$

The left hand side of this equation implicitly defines a linear operator $L : H^2_*(\Omega) \to H^2_*(\Omega)$ such that

$$(Lu,v)_{H^2_*(\Omega)} = (u,v)_{H^1_*(\Omega)} \qquad \forall v \in H^2_*(\Omega)\,.$$

The operator L is self-adjoint since

$$(Lu,v)_{H^2_*(\Omega)} = (u,v)_{H^1_*(\Omega)} = (v,u)_{H^1_*(\Omega)} = (Lv,u)_{H^2_*(\Omega)} = (u,Lv)_{H^2_*(\Omega)} \qquad \forall u,v \in H^2_*(\Omega)\,.$$

Moreover, by the compact embedding $H^2_*(\Omega) \subset H^1_*(\Omega)$ and the definition of L, the following implications hold:

$$u_n \rightharpoonup u \text{ in } H^2_*(\Omega) \implies u_n \to u \text{ in } H^1_*(\Omega) \implies \sup_{\|v\|_{H^2_*(\Omega)}=1} (u_n - u, v)_{H^1_*(\Omega)} \to 0$$
$$\implies \sup_{\|v\|_{H^2_*(\Omega)}=1} (L(u_n-u),v)_{H^2_*(\Omega)} \to 0 \implies Lu_n \to Lu \text{ in } H^2_*(\Omega)$$

which shows that L is also compact. Then, from the theory of linear compact self-adjoint operators, we know that L admits a sequence of eigenvalues and the corresponding eigenfunctions form an Hilbertian basis of $H^2_*(\Omega)$. This proves the last part of Theorem 4.

9 Proof of Theorem 5

The proof of Theorem 5 is based on the following result:

Theorem 11. *Let H be a Hilbert space and let $J \in C^1(H,\mathbb{R})$. Suppose that J satisfies the following assumptions:*

(i) J satisfies the Palais-Smale condition (PS): if u_n is a sequence such that $J(u_n)$ is bounded and $J'(u_n) \to 0$, then it has a converging subsequence.

(ii) J is bounded below.

(iii) J is twice differentiable for $u=0$, and $J''(0)$ has Morse index k.

(iv) J is even.

Then J has at least $2k$ pairs of nontrivial critical points $\pm u_j$, $j=1,...,k$.

Proof. This theorem is a variant of a classical result in critical point theory. For example, it can be deduced from Theorem 5.2.23, p.369 in [5]. Let us check that all the assumptions of this theorem are satisfied. Using the notation of [5], we take $a = \inf J$ and the condition $a < J(0)$ is satisfied. Let us check assumption (i). Take $E \subset H$ to be the space spanned by the eigenfunctions corresponding to the negative eigenvalues of the self-adjoint operator defined by the bilinear form $J''(0)[u,v]$. Then there exist $b < 0 = J(0)$ and $\rho > 0$ such that

$$\sup_{u \in E \cap \partial B_\rho} J(u) \le b \qquad (B_\rho := \{u \in H;\ \|u\| < \rho\})\,.$$

Assumption (ii) is verified taking $F = \{0\}$. Assumption (iii) is trivially verified. The conclusion of Theorem 5.2.23 in [5] is that there are at least $\dim E - \dim F$ couples of nontrivial critical points of J. Since $\dim E = \{\text{Morse index of } J''(0)\}$ and $\dim F = 0$, the conclusion of the theorem follows. □

In order to apply Theorem 11 to the functional $\mathcal{E}_0(u)$, we need the following two lemmas:

Lemma 12. *The functional $\mathcal{E}_f(u)$ satisfies (PS) on $H^2_*(\Omega)$.*

Proof. Let u_n be a sequence such that $\mathcal{E}_f(u_n)$ is bounded and $\mathcal{E}_f(u_n) \to 0$. Then we have that

$$\begin{aligned} M &\geq \mathcal{E}_f(u_n) = \frac{1}{2}\|u_n\|^2_{H^2_*(\Omega)} - P\int_\Omega \left(\sqrt{1+|\nabla u_n|^2}-1\right)dxdy - \langle f,u\rangle \\ &\geq \frac{1}{2}\|u_n\|^2_{H^2_*(\Omega)} - P\int_\Omega (1+|\nabla u_n|)\,dxdy - \|f\|_{\mathcal{H}}\|u_n\|_{H^2_*(\Omega)} \\ &\geq \frac{1}{2}\|u_n\|^2_{H^2_*(\Omega)} - 2\pi\ell P - \sqrt{2\pi\ell P}\sqrt{\int_\Omega |\nabla u_n|^2 dxdy} - \|f\|_{\mathcal{H}}\|u_n\|_{H^2_*(\Omega)} \\ &\geq \frac{1}{2}\|u_n\|^2_{H^2_*(\Omega)} - 2\pi\ell P - \sqrt{2\pi\ell P\Lambda}\|u_n\|_{H^2_*(\Omega)} - \|f\|_{\mathcal{H}}\|u_n\|_{H^2_*(\Omega)} \end{aligned} \tag{65}$$

Hence $\|u_n\|_{H^2_*(\Omega)}$ is bounded and u_n converges weakly to some $\bar{u} \in H^2_*(\Omega)$ (up to a subsequence). We need to prove that this convergence is strong. The condition $\mathcal{E}_f(u_n) \to 0$ can be rewritten as follows:

$$Lu_n + P\nabla\cdot\left(\frac{\nabla u_n}{\sqrt{1+|\nabla u_n|^2}}\right) - f = \chi_n$$

where $L : H^2_*(\Omega) \to \mathcal{H}$ is the linear operator defined by

$$\langle Lu, v\rangle = (u,v)_{H^2_*(\Omega)}$$

and χ_n is a sequence in $\mathcal{H}$ converging to 0. Since L is positive definite, it is invertible. The operator

$$u \mapsto \nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is continuous from $H^2_*(\Omega)$ to $L^2(\Omega)$ and hence it is compact from $H^2_*(\Omega)$ to $\mathcal{H}$. Then, the sequence $\frac{\nabla u_n}{\sqrt{1+|\nabla u_n|^2}}$ converges strongly in $\mathcal{H}$ (up to a subsequence). Then

$$u_n = L^{-1}\left[f - P\nabla\cdot\left(\frac{\nabla u_n}{\sqrt{1+|\nabla u_n|^2}}\right) + \chi_n\right]$$

converges strongly in $H^2_*(\Omega)$. □

Lemma 13. *The functional $\mathcal{E}_f(u)$ is bounded below and coercive.*

Proof. By (65) we have that

$$\mathcal{E}_f(u) \geq \frac{1}{2}\|u_n\|^2_{H^2_*(\Omega)} - a\|u_n\|_{H^2_*(\Omega)} - 2\pi\ell P \geq -\frac{1}{2}a^2 - 2\pi\ell P$$

where $a = \sqrt{2\pi\ell P\Lambda} + \|f\|_{\mathcal{H}}$. Thus $\mathcal{E}_f(u)$ is bounded below and coercive. □

We are now ready to prove Theorem 5. We need to show that the functional $\mathcal{E}_f(u)$ satisfies all the assumptions of the functional J in Theorem 11. It is straightforward to check that $\mathcal{E}_0 \in C^1(H^2_*(\Omega),\mathbb{R})$. By Lemmas 12 and 13 we have that J satisfies assumptions (i) and (ii). Assumption (iii) follows from (33) and the definition (26) of Λ_k. Finally assumption (iv) follows from the fact that we have $f = 0$.

10 Proof of Theorems 6 and 8

If $P \le \Lambda$, then the quadratic form

$$\|u\|^2_{H^2_*(\Omega)} - P\|u\|^2_{H^1_*(\Omega)}$$

is nonnegative definite; moreover, by (29), the term

$$\int_\Omega N(|\nabla u|)dxdy$$

is strictly convex and continuous. Hence the energy $\mathcal{E}_0(u)$ defined by (27) is a coercive strictly convex functional so that it has a unique minimum. This proves the first point of Theorem 6.

If $P > \Lambda$, by Lemma 13, the functional $\mathcal{E}_f$ preserves the coerciveness and, since it satisfies (PS) by Lemma 12, it has a minimizer. Hence (30) admits at least a weak solution $u \in H^2_*(\Omega)$. This proves the second point of Theorem 6.

For the multiplicity result (when $P > \Lambda$) we use a perturbation argument. By Theorem 5 we know that $\mathcal{E}_0(u)$ has two absolute minimum points $\pm\bar{u} \neq 0$. Then any small linear perturbation of $\mathcal{E}_0$ has a local minimum in a neighborhood of both $\pm\bar{u}$. Whence, if $\|f\|_{\mathcal{H}}$ is sufficiently small, the functional $\mathcal{E}_f$ defined by $\mathcal{E}_f(u) = \mathcal{E}_0(u) - \langle f, u\rangle$ admits two local minima in a neighborhood of $\pm\bar{u}$. These local minima, which we name v_1 and v_2, are the first two solutions of (30). A minimax procedure then yields an additional mountain-pass solution. More precisely, consider the class of all the continuous paths connecting v_1 and v_2:

$$\Gamma := \Big\{p \in C^0([0,1], H^2_*(\Omega));\ p(0) = v_1,\ p(1) = v_2\Big\}\,.$$

Since the functional $\mathcal{E}_f$ satisfies the Palais-Smale condition, the mountain-pass principle by Ambrosetti-Rabinowitz [2] guarantees that the level

$$\min_{p\in\Gamma}\ \max_{t\in[0,1]}\ \mathcal{E}_f\big(p(t)\big)\ >\ \max\Big\{\mathcal{E}_f(v_1), \mathcal{E}_f(v_2)\Big\}$$

is a critical level for $\mathcal{E}_f$; this yields a third solution to (30) and concludes the proof of Theorem 6.

The proof of Theorem 8 is similar. The quadratic part of the energy functional is the same. The remainder part is convex and all the above arguments can be slightly modified accordingly.

11 Proof of Theorem 7

If $P \le \Lambda$, then by taking $v = u$ in (40) and by using (22), we obtain

$$\left(1 - \frac{P}{\Lambda}\right)\|u\|^2_{H^2_*(\Omega)} + S\|u\|^4_{H^1_*(\Omega)} \le \|u\|^2_{H^2_*(\Omega)} - P\|u\|^2_{H^1_*(\Omega)} + S\|u\|^4_{H^1_*(\Omega)} = 0\,.$$

This shows that $u_0 = 0$ is the unique solution of (40). It is the absolute minimum of the coercive functional $\mathcal{E}_0$. The proof of Theorem 7 in the case $P \in (\Lambda_0, \Lambda_1]$ is therefore complete.

Assume now that $P \in (\Lambda_k, \Lambda_{k+1}]$ for some $k \ge 1$. Obviously, $u_0 = 0$ solves (40) for any S and P and the question is then whether u_0 is the only solution. In fact, u may be a nontrivial solution of (40) if and only if $-\delta(u)$ equals an eigenvalue of (23) (see (26)):

$$\Lambda_j = P - S\int_\Omega |\nabla u|^2\, dxdy$$

for some $j \ge 1$. Since $\Lambda_k < P \le \Lambda_{k+1}$ this is possible only for $j = 1, ..., k$ in which case

$$\|u\|^2_{H^1_*(\Omega)} = \frac{P - \Lambda_j}{S}\,. \tag{66}$$

This proves that (41) are the only solutions of (40). If some Λ_j has multiplicity larger than 1, this proves the existence of infinitely many solutions being eigenfunctions corresponding to Λ_j and satisfying (66).

Let us compute the energies of the solutions. By (66), and recalling that u_j solves (23) with $\Lambda = \Lambda_j$, we get

$$\|u_j\|^2_{H^1_*(\Omega)} = \frac{P-\Lambda_j}{S}\,, \qquad \|u_j\|^2_{H^2_*(\Omega)} = \frac{\Lambda_j(P-\Lambda_j)}{S}\,.$$

A simple computation then yields (42).

Finally, let us prove (43). We first remark that, in view of (23), any eigenfunction $\overline{w}_j$ corresponding to the eigenvalue Λ_j satisfies

$$\Lambda_j\,(\overline{w}_j, v)_{H^1_*(\Omega)} = (\overline{w}_j, v)_{H^2_*(\Omega)} \quad \forall v \in H^2_*(\Omega)\,; \tag{67}$$

in particular, we have

$$\|\overline{w}_j\|_{H^1_*(\Omega)} = 1\;\forall j\,, \quad v = \sum_{i=1}^{\infty} \alpha_i \overline{w}_i \quad \Longrightarrow \quad \|v\|^2_{H^2_*(\Omega)} = \sum_{i=1}^{\infty} \Lambda_i \alpha_i^2\,, \quad \|v\|^2_{H^1_*(\Omega)} = \sum_{i=1}^{\infty} \alpha_i^2\,. \tag{68}$$

Let u_j ($j = 1, ..., k$) be a solution of (40), see (41), and let $v \in H^2_*(\Omega)$. For all $t \in \mathbb{R}$ we compute $\mathcal{E}_0(u_j + tv) - \mathcal{E}_0(u_j)$; since u_j solves (40) we obtain

$$\begin{aligned}
\mathcal{E}_0(u_j{+}tv){-}\mathcal{E}_0(u_j) &= t^2\left[\frac{\|v\|^2_{H^2_*(\Omega)}}{2}{+}\left(\frac{S}{2}\|u_j\|^2_{H^1_*(\Omega)} - \frac{P}{2}\right)\|v\|^2_{H^1_*(\Omega)}{+}S(u_j,v)^2_{H^1_*(\Omega)}\right]{+}o(t^2) \quad \text{as } t{\to}0\\
\text{by (41)-(67)} \quad &= t^2\left[\frac{\|v\|^2_{H^2_*(\Omega)}}{2}{-}\frac{\Lambda_j}{2}\|v\|^2_{H^1_*(\Omega)}{+}S(u_j,v)^2_{H^1_*(\Omega)}\right]{+}o(t^2) \quad \text{as } t{\to}0.
\end{aligned} \tag{69}$$

In turn, if v is as in (68) and we apply the implications there, then (69) implies

$$\begin{aligned}
\mathcal{E}_0(u_j + tv) - \mathcal{E}_0(u_j) &= t^2\left[\frac{1}{2}\sum_{i=1}^{\infty}\Lambda_i\alpha_i^2{-}\frac{\Lambda_j}{2}\sum_{i=1}^{\infty}\alpha_i^2{+}(P-\Lambda_j)\alpha_j^2\right]{+}o(t^2) \quad \text{as } t{\to}0\\
&= t^2\left[\frac{1}{2}\sum_{i=j+1}^{\infty}(\Lambda_i{-}\Lambda_j)\alpha_i^2{+}(P{-}\Lambda_j)\alpha_j^2{-}\frac{1}{2}\sum_{i=1}^{j}(\Lambda_j{-}\Lambda_i)\alpha_i^2\right]{+}o(t^2) \quad \text{as } t{\to}0.
\end{aligned}$$

This shows that the second derivative of $\mathcal{E}_0(u)$ at $u = u_j$ is a linear operator which is negative definite on a subspace of finite dimension $j-1$ and positive definite on its orthogonal subspace (which has codimension $j-1$). This proves (43) for $j = 1, ..., k$.

For u_0, the same argument yields

$$\mathcal{E}_0(u_0 + tv) - \mathcal{E}_0(u_0) = \mathcal{E}_0(tv) = \left(\frac{1}{2}\|v\|^2_{H^2_*(\Omega)} - \frac{P}{2}\|v\|^2_{H^1_*(\Omega)}\right)t^2 + \frac{S}{4}\|v\|^4_{H^1_*(\Omega)}\,t^4\,.$$

Therefore, $t \mapsto \mathcal{E}_0(tv)$ has a local maximum at $t = 0$ if and only if $\|v\|^2_{H^2_*(\Omega)} < P\|v\|^2_{H^1_*(\Omega)}$, a case which occurs for $v \in \mathrm{span}\{\overline{w}_1, ..., \overline{w}_k\}$. This proves (43) also for u_0.

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