Some results about stationary Navier-Stokes equations with a pressure-dependent viscosity

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Abstract

We consider the stationary Navier-Stokes equations with a pressure-dependent viscosity. For "almost conservative" external forces, we prove by application of the local inversion theorem that the homogeneous Dirichlet problem admits a unique regular solution.

1 Introduction

In this paper we study the stationary problem

$$\begin{cases} -\nabla\cdot[\eta(p)(\nabla u+\nabla^T u)]+\nabla p+(u\cdot\nabla)u=f \qquad \text{in} \quad \Omega \\ \nabla\cdot u=0 \qquad \text{in} \quad \Omega \\ u=0 \qquad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where the unknowns are the vector function u and the scalar function p and Ω is an open bounded set of $\mathbb{R}^n$ $(n\geq 2)$ with smooth boundary $\partial\Omega$. This problem represents Navier-Stokes equations with a pressure-dependent viscosity: u is the velocity of the fluid, p its pressure, η its viscosity and f an external force acting on the fluid. Our research is motivated by physical experiments, indeed the viscosity η of real fluids may depend on the pressure and on the temperature: this dependence is best represented by Eyring's equation [1, 5]

$$\eta(p,T)=a\sqrt{T}\cdot\exp\left(\frac{b+cp}{T}\right) \tag{2}$$

where T denotes the temperature of the fluid and $a,b,c>0$ are constants depending on the nature of the fluid; we deal with a fluid at constant temperature so that the viscosity η only depends on p, i.e. $\eta=\eta(p)$. We require the function η to satisfy the realistic assumptions

$$\eta\in C^{m+3}(\mathbb{R}) \qquad \text{and} \qquad \inf_{x\in\mathbf{R}}\eta(x)=\eta_0>0\ , \tag{3}$$

where $m \in \mathbb{N}$ will be chosen depending on the regularity we expect for the solution of (1). To remove an indeterminacy already present in the classical Navier-Stokes equations we also need a condition on p: more precisely, we will assume that the mean value of p over Ω is known.

Navier-Stokes equations with a pressure-dependent viscosity were first studied by Renardy [6] in the 3D case; Renardy remarks that if (2) does not break down for large pressures we cannot even guarantee the existence of a pressure p for a prescribed velocity field u and body force f: under the assumptions that η is sublinear at infinity, that $|\eta'|_\infty = \eta'_\infty < +\infty$ an existence and uniqueness result for the evolution equation is proved. Moreover, Renardy remarks that the stationary equations (1) may lose their ellipticity unless a condition on the eigenvalues of the tensor $(\nabla u + \nabla^T u)$ is assumed; more precisely, it is necessary to assume that

$$\text{the eigenvalues of} \quad (\nabla u + \nabla^T u) \quad \text{are strictly less than} \quad \frac{1}{\eta'_\infty} \,. \tag{4}$$

In the stationary case, the main result of Renardy is an existence and uniqueness result for p (with given mean value) to satisfy (1) in a suitable weak sense if the velocity u is known and (3) (4) hold. This result is useful in view of the elimination of the pressure p in the equation: indeed to eliminate the pressure in (1) we cannot make use of Hodge projection (see [7]) as for the classical problem and we must solve a nonlinear elliptic equation in divergence form.
In a subsequent paper [2] a similar result is proved (in the case $n = 3$) by taking stronger assumptions on η, by weakening the regularity assumptions on u and f and by replacing (4) with

$$|\nabla u|_\infty < \frac{1}{2\eta'_\infty \sqrt{6}} \;;$$

the existence of a unique function p (with given mean value) satisfying (1) in a suitable weak sense is then proved. In [2] a converse result is also proved: if the pressure p is known and belongs to a suitable neighborhood of its mean value then there exists a unique u satisfying (1) in a "complementary" weak sense; the proofs of these results involve Helmholtz-Weyl decomposition of the space $\mathbf{L}^2(\Omega)$.
The aim of this paper is to prove, by application of the local inversion theorem, the existence of a unique regular solution to (1) when the external force f is "almost conservative" in a suitable sense: this result will be used in a forthcoming paper [3] to prove local existence and uniqueness for the evolution problem under the only assumption (3) on the viscosity.

2 Statements of the results

We assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set satisfying

$$\partial\Omega \in C^{m+2} \tag{5}$$

for some integer m. With bold capital letters ($\mathbf{L}^2$, $\mathbf{H}^1$, $\mathbf{V}$,...) we denote functional spaces of vector functions and with usual capital letters (L^2, H^1,...) we denote functional spaces of scalar functions.

To simplify notations we set $L^2 := L^2(\Omega)$, $\mathbf{H}^1 := \mathbf{H}^1(\Omega)$,... With $W^{m,s}$ we represent Sobolev spaces of functions with generalized derivatives up to order m in L^s, with $\|\cdot\|_m$ we denote the corresponding norm and with $W_0^{m,s}$ the $W^{m,s}-$closure of the space C_c^∞ of smooth functions with compact support in Ω; $H^m := W^{m,2}$ represent the Hilbertian Sobolev spaces. Given Banach spaces $\mathbf{X}, \mathbf{Y}$, we denote by $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ the space of linear continuous operators from $\mathbf{X}$ to $\mathbf{Y}$ and by $\|\cdot\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})}$ its norm.

We also need the spaces arising from Helmholtz-Weyl decomposition of the Hilbert space $\mathbf{L}^2$ (see [8])

$$\mathbf{G} := \{f \in \mathbf{L}^2;\ \nabla \cdot f = 0,\ \gamma_n f = 0\} \qquad \mathbf{G}^\perp := \{f \in \mathbf{L}^2;\ \exists g \in H^1,\ f = \nabla g\}$$
$$\mathbf{V} := \{f \in \mathbf{H}_0^1;\ \nabla \cdot f = 0\}$$

where γ_n denotes the normal trace operator; the space $\mathbf{V}$ is a Hilbert space when endowed with the scalar product $(u,v)_\mathbf{V} = (\nabla u, \nabla v)_{\mathbf{L}^2}$. It is well-known that $\mathbf{L}^2 = \mathbf{G} \oplus \mathbf{G}^\perp$: we denote by $\mathcal{P}$ (resp. $\mathcal{Q}$) the orthogonal projectors of $\mathbf{L}^2$ onto $\mathbf{G}$ (resp. $\mathbf{G}^\perp$). The projections $\mathcal{P}f$ and $\mathcal{Q}f$ of a function $f \in \mathbf{L}^2$ are determined by solving the homogeneous Dirichlet problem for a Poisson equation and a Neumann problem for Laplace equation. Recall that $\mathcal{P}$ is a linear continuous operator from $\mathbf{W}^{m,s}$ onto $\{f \in \mathbf{W}^{m,s};\ \nabla \cdot f = 0,\ \gamma_n f = 0\}$.

As in [6] we assume that the mean value of p over Ω is given, say $\bar{p}$: without loss of generality we take $\bar{p} = 0$; assume that

$$m \in \mathbb{N} \qquad \text{and} \qquad s > 1 \qquad \text{satisfy} \qquad (m+1)s > n \tag{6}$$

in order to have the imbedding $\mathbf{W}^{m+1,s} \subset \mathbf{L}^\infty$. Consider the spaces

$$\overline{W}^{m+1,s} := \left\{ g \in W^{m+1,s};\ \int_\Omega g(x)\ dx = 0 \right\}$$

and $\mathbf{U}_{m,s} := \mathbf{W}^{m+2,s} \cap \mathbf{V}$ normed with the $\mathbf{W}^{m+2,s}$-norm; the space $\mathbf{X}_{m,s} := \mathbf{U}_{m,s} \times \overline{W}^{m+1,s}$ is a Banach space when endowed with the norm

$$\forall (u,p) \in \mathbf{X}_{m,s} \qquad \|(u,p)\|_\mathbf{X} = \|u\|_{m+2,s} + \|p\|_{m+1,s}\ .$$

The space $\mathbf{X}_{m,s}$ is the space where we will seek the solutions (u,p) of (1); to simplify the notations we omit the indices m and s on the spaces $\mathbf{X}, \mathbf{U}$. Set also $\mathbf{Y} := \mathbf{W}^{m,s}$, $\|\cdot\|_\mathbf{Y} = \|\cdot\|_m$. Finally, let us introduce the operator $\Phi : \mathbf{X} \to \mathbf{Y}$ (see Lemma 3) defined by

$$\Phi(u,p) = -\nabla \cdot [\eta(p)(\nabla u + \nabla^T u)] + \nabla p + (u \cdot \nabla) u\ . \tag{7}$$

Definition
Let $f \in \mathbf{Y}$, we say that (u,p) is a *solution* of (1) if $\Phi(u,p) \in \mathbf{Y}$ and $\Phi(u,p) = f$ a.e. in Ω.

Making use of an inversion argument we will prove

Theorem 1 *Assume (3) (5) (6) and let* $\psi \in \overline{W}^{m+1,s}$*; then there exists a constant* $R = R(\psi) > 0$ *such that if* $f \in \mathbf{Y}$ *satisfies* $\|f - \nabla\psi\|_{\mathbf{Y}} \leq R$*, then (1) admits a unique solution* $(u,p) \in \mathbf{X}$ *in a suitable* $\mathbf{X}$*-neighborhood of* $(0,\psi)$*. Moreover, the map* $f \mapsto (u,p)$ *is continuous from* $\mathbf{Y}$ *to* $\mathbf{X}$*. In particular, there exists* $\bar{R} > 0$ *such that if* $\|f\|_{\mathbf{Y}} \leq \bar{R}$*, then (1) admits a unique solution in a suitable* $\mathbf{X}$*-neighborhood of* $(0,0)$*.*

In particular, we have

Corollary 1 *Assume (3) (5) (6) and let* $f \in \mathbf{Y}$ *be such that* $\mathcal{P}f = 0$ *(i.e.* $f \in \mathbf{G}^\perp$*); then (1) admits a unique solution* $(u,p) \in \mathbf{X}$ *given by*

$$u \equiv 0 \qquad p = g$$

where g *is the zero mean value potential of* f *(i.e.* $g \in \overline{W}^{m+1,s}$*,* $\nabla g = f$*).*

Therefore, for conservative forces f the classical problem with constant viscosity and (1) have the same solutions: the equations become the well-known equation of the statics of the fluids and this fact shows that the model described by (1) is consistent.

3 Proof of Theorem 1

We begin with two technical results:

Lemma 1 *Assume (3) (5) (6); then:*
(i) there exists a constant $\omega = \omega(\eta_0, n, \Omega) > 0$ *such that if* $f \in \mathbf{Y}$ *and* $(u,p) \in \mathbf{X}$ *is a solution of (1) then* $\|u\|_{\mathbf{V}} \leq \omega \|\mathcal{P}f\|_{\mathbf{Y}}$
(ii) for all $p \in \overline{W}^{m+1,s}$ *the bilinear form on* $\mathbf{V}$

$$\forall u, v \in \mathbf{V} \qquad [u,v]_p := \int_\Omega \eta(p)(\nabla u + \nabla^T u) : \nabla v$$

defines a scalar product which induces a norm equivalent to $\|\cdot\|_{\mathbf{V}}$*.*

Proof. Multiply (1) by u, integrate by parts to obtain

$$\int_\Omega \eta(p)(\nabla u + \nabla^T u) : \nabla u = \int_\Omega uf$$

and remark that $\int_\Omega uf = \int_\Omega u\mathcal{P}f$ by the imbedding $\mathbf{V} \subset \mathbf{G}$. Next, note that

$$(\nabla u + \nabla^T u) : \nabla u = \frac{1}{2}(\nabla u + \nabla^T u) : (\nabla u + \nabla^T u) \geq 0 \tag{8}$$

and that, by the divergence Theorem,

$$\int_\Omega \nabla u : \nabla^T u = \int_\Omega \nabla \cdot [(u \cdot \nabla)u] = \int_{\partial\Omega} \gamma_n[(u \cdot \nabla)u] = 0 \ ;$$

hence, by (3)

$$\eta_0 \int_\Omega |\nabla u|^2 \le \int_\Omega \eta(p)(\nabla u + \nabla^T u) : \nabla u \ . \tag{9}$$

Therefore, we have

$$\eta_0 \int_\Omega |\nabla u|^2 \le \int_\Omega u \mathcal{P} f$$

and (i) follows by Schwarz inequality and the imbedding $\mathbf{Y} \subset \mathbf{H}^{-1}$.
The bilinear form $[u,v]_p$ is clearly symmetric and by (9) it is also a positive form. By the imbedding $\overline{W}^{m+1,s} \subset L^\infty$ and by (3) we infer that

$$\forall p \in \overline{W}^{m+1,s} \quad \exists C_p > 0 \qquad \text{such that} \qquad C_p \ge \eta(p) \ge \eta_0 \quad \text{for a.e. } x \in \Omega \ ;$$

hence, by (8) we get

$$\forall u \in \mathbf{V} \qquad \eta_0 \int_\Omega |\nabla u|^2 \le \int_\Omega \eta(p)(\nabla u + \nabla^T u) : \nabla u \le C_p \int_\Omega |\nabla u|^2 \ ,$$

which proves (ii). □

The following lemma is obtained by slight modifications of well-known results about the classical Stokes problem:

Lemma 2 *Assume (3) (5) (6) and let* $\psi \in \overline{W}^{m+1,s}$*; then, for all* $g \in \mathbf{Y}$ *there exists a unique solution* $(v,q) \in \mathbf{X}$ *of the equation*

$$-\nabla \cdot \Big[\eta(\psi)(\nabla v + \nabla^T v)\Big] + \nabla q = g \qquad in \quad \Omega \ . \tag{10}$$

Proof. If $\eta(\psi) \equiv \eta$ the result is well-known, see [4].
If $\eta = \eta(\psi)$, by Lemma 1 the bilinear form $[\cdot,\cdot]_\psi$ is coercive: hence, by standard estimates for elliptic operators one has that if (10) admits a solution $(v,q) \in \mathbf{X}$, then $\|(v,q)\|_{\mathbf{X}} \le c\|g\|_{\mathbf{Y}}$. Build a homotopy connecting (10) and the classical Stokes problem: then Proposition 6.1 in [9] gives the result. □

Next, we prove some regularity properties of Φ:

Lemma 3 *Assume (3) (5) (6) and let* Φ *be the operator defined in (7). Then* $\Phi \in C(\mathbf{X},\mathbf{Y})$.

Proof. Given $(u,p),\ (v,q) \in \mathbf{X}$ we have

$$\begin{aligned}
\|\Phi(u,p) - \Phi(v,q)\|_{\mathbf{Y}} \ &\le \ \|[\eta(p)-\eta(q)](\nabla u + \nabla^T u)\|_{m+1} + \|\eta(q)[\nabla(u-v) + \nabla^T(u-v)]\|_{m+1} \\
&\quad + \|p-q\|_{m+1} + \|((u-v)\cdot\nabla)u\|_m + \|(v\cdot\nabla)(u-v)\|_m \\
&\le \ c\|[\eta(p)-\eta(q)]\|_{m+1}\|u\|_{m+2} + c\|\eta(q)\|_{m+1}\|u-v\|_{m+2} \\
&\quad + \|p-q\|_{m+1} + \|u-v\|_{m+1}(\|u\|_{m+1} + \|v\|_{m+1}).
\end{aligned} \tag{11}$$

If $q \to p$ in $W^{m+1,s}$, then we can show that $\eta(q) \to \eta(p)$ in $W^{m+1,s}$. Thus from (11) we obtain the continuity of Φ. □

Lemma 4 *The Fréchet-derivative Φ' of Φ is continuous at any point $(0, \psi), \psi \in \overline{W}^{m+1,s}$.*

Proof. By a straightforward calculation one verifies that for any $(u, p) \in \mathbf{X}$ the operator Φ has a Fréchet-derivative $\Phi'(u, p) : \mathbf{X} \to \mathbf{Y}$ defined by $\Phi'(u, p)[v, q] = \Phi_u(u, p)[v] + \Phi_p(u, p)[q]$ where

$$\Phi_u(u,p)[v] = -\nabla \cdot [\eta(p)(\nabla v + \nabla^T v)] + (u \cdot \nabla)v + (v \cdot \nabla)u$$
$$\Phi_p(u,p)[q] = [\mathbf{I} - \eta'(p)(\nabla u + \nabla^T u)]\nabla q + [-\eta'(p)\Delta u - \eta''(p)(\nabla u + \nabla^T u)\nabla p]q \ .$$

Then, we have

$$\begin{aligned} \|\Phi_u(u,p)[v] - \Phi_u(0,\psi)[v]\|_{\mathbf{Y}} &\leq \|\nabla \cdot [(\eta(p) - \eta(\psi))(\nabla v + \nabla^T v)]\|_m + \|(u \cdot \nabla)v\|_m + \|(v \cdot \nabla)u\|_m \\ &\leq c\|\eta(p) - \eta(\psi)\|_{m+1}\|v\|_{m+2} + c\|u\|_{m+2}\|v\|_{m+2} \ . \end{aligned}$$

Thus we obtain

$$\|\Phi_u(u,p) - \Phi_u(0,\psi)\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})} \leq c\|\eta(p) - \eta(\psi)\|_{m+1} + c\|u\|_{m+2}$$

which goes to 0 as $\|u, p - \psi\|_{\mathbf{X}} \to 0$. For Φ_p we proceed similarly. □

Proof of Theorem 1. Given $\psi \in \overline{W}^{m+1,s}$, we have $\Phi(0, \psi) = \nabla\psi$. Φ is continuous on $\mathbf{X}$ and Φ' is continuous at $(0, \psi)$ because of Lemmata 3 and 5. Since $\Phi'(0, \psi)[v, q] = -\nabla \cdot [\eta(\psi)(\nabla v + \nabla^T v)] + \nabla q$, $\Phi'(0, \psi)$ is an invertible operator from Lemma 2. Thus the result follows by applying the local inversion theorem. □

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