Old and new explanations of the Tacoma Narrows Bridge collapse

Gianni Arioli[1,2], Filippo Gazzola[1]

[1]*Department of Mathematics,* [2]*MOX, Politecnico of Milan, Italy*
E-mail: *gianni.arioli@polimi.it, filippo.gazzola@polimi.it*

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SUMMARY. We survey some explanations of the Tacoma Narrows Bridge collapse, which occurred in 1940: none of them is universally accepted. We then introduce a new mathematical model for the study of the dynamical behavior of suspension bridges and we show that it provides a realistic explanation of the Tacoma collapse.

1 INTRODUCTION

On November 7, 1940, the Tacoma Narrows Bridge (TNB) collapsed and a movie [1] testifies this amazing event. The wind was blowing at approximately 80 km/h and, without any intermediate stage, a violent destructive torsional oscillation started. These oscillations were already seen in other bridges before 1940, see Section 4.3 in [2] and Section 2 in [3]. Natural questions arising from the TNB collapse and from similar failures are the following [p.53,2]:

- how could a span designed to withstand 161 km/h winds succumb under a wind of less than half that velocity?
- how could horizontal wind forces be translated into dynamic vertical and torsional motion?

The answer to the first question appears fairly simple. The bridge was ready to withstand 161 km/h wind provided that the oscillation would have been longitudinal. But since unexpected torsional oscillations appeared, this considerably lowered the critical speed of the wind. The Report [4] considers *the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*. Therefore, the two above questions reduce to the following:

why do torsional oscillations appear suddenly in suspension bridges?

Many attempts to explain the TNB collapse have been made but, after 73 years, a full explanation of the reasons of the collapse is not available, see again [2]. In the next section we survey some of these attempts and we explain why they fail to answer to the above question. Then we recall the model suggested in [5], we complement it with some new numerical results, and we explain how it may be used to give a satisfactory answer.

2 PRIOR EXPLANATIONS OF THE TACOMA NARROWS BRIDGE COLLAPSE

2.1 *Structural failure*

A first natural explanation of the collapse is a mistake in the project. From the New York Times [6] we recall a comment by Andrew, chief engineer in charge of constructing the bridge, who claims that *the collapse probably was due to the fact that flat, solid girders were used along one side of the span. These girders, he said, caught the wind like a kite and caused the bridge to sway*. However, the video of the collapse does not show the TNB as a kite. Moreover, Leon Moisseiff, who was charged with the project was not considered guilty for the TNB failure: Steinman-Watson [7] wrote that *the span failure is not to be blamed on him; the entire profession shares in the responsibility. It is simply that the profession had neglected to combine, and apply in* *time, the knowledge of aerodynamics and of dynamic vibrations with its rapidly advancing knowledge of structural design*.

Several people attempted to justify the collapse with a structural failure. Delatte [8] suggests that *a contributing factor may have been slippage of a band that retained the cables*. Then, by invoking [p.226,9], he writes that *on November 7 a cable band slipped out of place at mid-span, and the motions became asymmetrical, like an airplane banking in different directions. The twisting caused metal fatigue, and the hangers broke like paper clips that had been bent too often*. This colorful description was not confirmed by subsequent studies.

In the preface of the Report [4] one reads that *the first failure was the slipping of the cable … this slipping probably initiated the torsional oscillations … these torsional movements caused breaking stresses at various points … further structural damage followed almost immediately*. This kind of "domino effect" was not analyzed further in the sequel and, a few years after the collapse, Steinman [10] wrote that the ties *permitted - not caused - the catastrophic oscillations that wrecked the structure*.

Steinman designed the Thousand Islands Bridge (1938) and the Deer Isle Bridge (1939), which had similar narrow and flexible structures as the TNB. As mentioned at [p.33,8], *both bridges exhibited substantial oscillations in the wind and required stiffening with cable ties. Vertical and torsional motions were both observed. The cables ties substantially reduced, but did not eliminate, the motions, and were thought to be an adequate repair*. Hence, the phenomenon observed at the TNB is present in similar bridges and the collapse cannot be attributed to a structural failure.

Steinman [11] claimed that [4] *leaves many questions unanswered. It does not tell what combinations of cross-sections produce aerodynamic instability, how aerodynamic instability can be reasonably predicted or readily tested, nor how it can be prevented*. His final observation is that *it is more scientific to eliminate the cause than to build up the structure to resist the effect*.

2.2 *External resonance*

In an article appeared in the New York Times [12] a couple of days after the collapse, one may read *like all suspension bridges, that at Tacoma both heaved and swayed with a high wind. It takes only a tap to start a pendulum swinging. Time successive taps correctly and soon the pendulum swings with its maximum amplitude. So with the bridge. What physicists call resonance was established, with the result that the swaying and heaving exceeded the limits of safety*.

One of the conclusions of the Federal Report [p.128,4] is that *resonance with alternating periodic eddies (sometimes called Karman vortices) has been mentioned as a possible inducement to large amplitude oscillations*. But most people believe that the Report does not explain how the wind, random in nature, might produce a precise periodic impulse. The mathematicians Lazer-McKenna [13] point out that *the phenomenon of linear resonance is very precise. Could it really be that such precise conditions existed in the middle of the Tacoma Narrows, in an extremely powerful storm*? The physicists Green-Unruh [14] mention that *making the comparison to a forced harmonic oscillator requires that the wind generates a periodic force tuned to the natural frequency of the bridge*. Among engineers, Scanlan [15] writes *others have added to the confusion. A recent mathematics text, for example, seeking an application for a developed theory of parametric resonance, attempts to explain the Tacoma Narrows failure through this phenomenon*. Moreover, Billah-Scanlan [16] make a fool of physics textbooks who attempt to explain the TNB collapse with resonance.

History may explain why the TNB collapse was attributed to resonance. Built in 1826, the Broughton Suspension Bridge collapsed in 1831 due to mechanical resonance induced by troops marching over the bridge in step. Since then, all troops "break step" when crossing a bridge. The probability that the step frequency of a troop coincides exactly with a natural frequency of a bridge is zero, but if these frequencies almost coincide then, unconsciously, the step of the humans tends to approach a natural frequency of the structure. Built in 1839, the Angers Suspension Bridge collapsed in 1850 while a battalion of soldiers was marching across it. The soldiers had been ordered to break step and to space themselves farther apart than normal. But the battalion arrived during a thunderstorm when the wind was making the bridge oscillate and their efforts to match the swaying and keep their balance has caused them to involuntarily march with the same cadence, contributing to the resonance. A related but slightly different behavior was observed in June 2000, on the opening day of the London Millennium Bridge: the crowd streamed on it, the bridge started to sway from side to side and pedestrians fell spontaneously into step with the oscillations, thereby amplifying them, see [17]. In all these accidents the external forcing was periodic and precise, similar to one of the vibrating modes of the bridge. But this was not the case at the TNB.

Hence, an external resonance, intended as a perfect matching between the exterior wind and the parameters of the bridge, is not the culprit for the TNB collapse.

2.3 *Vortices*

Every oscillating structure has its own natural frequencies and resonance occurs if the excitation force acts periodically and with one of the natural frequencies. Due to the non-streamlined shape of the bridge, a possible candidate of the periodicity in the wind force was the vortex shedding. These wakes are accompanied by alternating low-pressure vortices on the downwind side of the body, the von Karman vortex street. As a consequence, the bridge would move towards the low-pressure zone, in an oscillating movement called vortex-induced vibration. If the frequency of vortex shedding matches the natural frequency of the bridge, then the structure resonates and oscillations may become self-sustaining. Von Karman claimed that the motion of the TNB was due to these vortices and that the von Karman street wake reinforced the already present oscillations and caused the center span to violent twist until the bridge collapse, see [p.31,8]. But, according to Scanlan [15], *some of the writings of von Karman leave a trail of confusion ... it can clearly be shown that the rhythm of the failure (torsion) mode has nothing to do with the natural rhythm of shed vortices following the Karman vortex street pattern*. And, indeed, the calculated frequency of a vortex caused by a 68 km/h wind is 1 Hz, whereas the frequency of the torsional oscillations measured by Farquharson was 0.2 Hz, see [p.120,8]. The conclusion on [p.122,16] is that *we see the flutter vortex trail as a consequence, not as a primary cause*.

Also Green-Unruh [11] claim that *the von Karman vortex street forms at a frequency determined by the geometry and the wind velocity. These vortices form independently of the motion and are not responsible for the catastrophic oscillations of the TNB*. Their own conclusion is similar, namely *vortices are also produced as a result of the body's motion*.

What remained unsolved for a long time was how vortices could be responsible for the wind-excited twisting motion. In 2000, Larsen [p.247,18] writes *the vortex street may cause limited torsion oscillations, but cannot be held responsible for divergent large-amplitude torsion oscillations*. Then Larsen [p.245,18] claims that *the key to the torsion instability mechanism is the formation and drift of large-scale vortices on the cross section. A discrete vortex simulation of the flow around a simplified model of the Tacoma Narrows section shape, in which the angle of attack changes stepwise from 0° to 10°, highlights the vortex dynamics involved*. Whence, it is claimed that the variation of the angle of attack creates an alternation of vortices characterized by the direction of rotation and the position above/below the roadway. These vortices are also due to the H-form of the cross section and may push up or pull down the endpoints of the cross section. The variation of the angles also generates extra energy that gives rise to higher amplitudes of torsional oscillations and the cross section oscillates in a self-sustaining motion. Although this theory may describe the self-exciting phenomenon and the increase of the width of torsional oscillations, it does not explain how the torsional oscillation starts. An explanation with stepwise angles of attack could be reliable only if the wind did not vary the angle of attack for a long time and, suddenly, it started varying the angle of attack creating a torsional motion. It also appears somehow obvious that if the wind is known to vary stepwise the angle of attack of about 10°, then the movement of the cross section of the roadway will be quite similar to a swing. But did the wind really vary stepwise the angle of attack on November 7, 1940? From [p.130,4] we quote *It is very improbable that resonance with alternating vortices plays an important role in the oscillations of bridges*.

Skeptic comments on the Larsen work were also made by Green-Unruh [14] who write *this analysis is somewhat incomplete given the data available* and claim that *the Larsen model does not adequately explain data at around 23 m/s*, which was the wind velocity the day of the collapse.

Green-Unruh pursue the Larsen explanation under three different aspects: they study how vortices drift near boundaries, how a vortex drifts near the trailing edge of the bridge, and the production of vortices at the leading edge. The conclusion in [14] contains several criticisms on their own work; they write *the detailed method through which the oscillatory behavior is established may require further details ... the range of wind speeds where the model is applicable has not been fully established. At the extreme high and low values, computational calculations become less reliable*. This is true in general: as long as a phenomenon is in a suitable range, any explanation is satisfactory. But the TNB behavior was not in a "reasonable range".

2.4 *Flutter theory*

Flutter is a self-feeding and potentially destructive vibration where aerodynamic forces on an object couple with a structure's natural mode of vibration to produce rapid periodic motion. Flutter may occur in any object within a fluid flow, under the conditions that a positive feedback occurs between the structure's natural vibration and the aerodynamic forces. The vibrational movement of the object increases an aerodynamic load, which in turn drives the object to move further. If the energy input by the aerodynamic excitation in a cycle is larger than that dissipated by the damping in the system, the amplitude of vibration increases, resulting in self-exciting oscillations.

While discussing oscillations in bridges, the physicist Rocard [p.185,19] attributes to Bleich [20] *to have pointed out the connection with the flutter speed of aircraft wings... He distinguishes clearly between flutter and the effect of the staggered vortices and expresses the opinion that two degrees of freedom (bending and torsion) at least are necessary for oscillations of this kind*. At pp.246-247 in [21] it is assumed that the bridge is subject to a natural steady state oscillating motion and the flutter speed is defined as follows: *with increasing wind speed the external force necessary to maintain the motion at first increases and then decreases until a point is reached where the air forces alone sustain a constant amplitude of the oscillation. The corresponding velocity is called the critical velocity or flutter speed*. The flutter speed is then described by noticing that *below the critical velocity V_c an exciting force is necessary to maintain a steady-state motion; above the critical velocity the direction of the force must be reversed (damping force) to maintain the steady state motion. In absence of such a damping force the slightest increase of the velocity above V_c causes augmentation of the amplitude*. Whence, if exceeded, the flutter speed may give rise to uncontrolled phenomena such as torsional oscillations. Rocard [p.101,19] claims that the main contribution of his own work is a *precise method of calculating the critical speed of wind for any given suspension bridge*. With the parameters of the TNB, his computations lead to a critical speed of wind coinciding with that of the day of the collapse, see [p.158,19]. As far as we are aware, Rocard's method to determine the flutter speed was not used in later plans. Further credit to [20] is given on [p.80,2] where one reads: *Bleich's work ... ultimately opened up a whole new field of study. Wind tunnel tests on thin plates suggested that higher wind velocities increased the frequency of vertical oscillation while decreasing that of torsional oscillation*. The conclusion is that *Bleich's work could not be used to explain the Tacoma Narrows Bridge collapse*.

While referring explicitly to the work by Bleich and Rocard, Billah-Scanlan [p.122,16] write that *another error accompanying many accounts has been the confusion of the phenomenon of bridge flutter with that of airplane wing flutter as though they were identical*. Billah-Scanlan also emphasize that *forced resonance and self-excitation are fundamentally different phenomena* and they claim that their work *demonstrates that the ultimate failure of the bridge was in fact related to an aerodynamically induced condition of self-excitation or "negative damping" in a torsional degree of freedom*. The negative damping together with the torsional degree of freedom caused the torsional flutter so that, as the roadway rotated, the wind force acting on the surface changed, when the bridge rotated back the forces pushed the bridge in the opposite direction. They claim that this negative damping effect and increase in rotation lead up to the torsional oscillation that caused the collapse of the bridge. But Larsen [p.244,18] writes that *Billah and Scanlan ... fail to connect the vortex pattern to the shift of apparent section damping from positive to negative, which signifies the onset of torsional instability*. We think that self-excited oscillations may appear only if some oscillations already exist, but what creates the "first" torsional oscillation?

3 APPEARANCE OF TORSIONAL OSCILLATIONS IN ISOLATED SYSTEMS

Consider a suspension bridge subject to gravity and to the elastic restoring forces due to the action of the hangers. We model the bridge by considering a number n of parallel rods, representing the cross sections of the roadway whose endpoints are linked to the hangers. Each rod interacts with the two nearest neighbors rods with linear attractive forces representing resistance to longitudinal and torsional stretching. This model is inspired to the celebrated Fermi-Pasta-Ulam model [22]. In Figure 1 we represent the model, the red cross sections being the rods linked to the hangers, which behave as nonlinear springs, and the grey part of the roadway being the connections between rods and behaving like an elastic membrane.

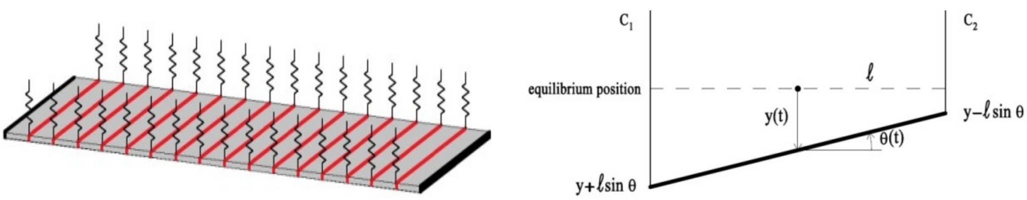

Figure 1. A discretized suspension bridge (left); vertical and torsional movements of a rod (right).

Each cross section is modeled as a rod, of mass m and length 2ℓ, subject to the forces exerted by the two lateral hangers C_1 and C_2, see again Figure 1: θ is the angle of deflection from horizontal, y is the downwards displacement from equilibrium of the barycenter. The forces are denoted by $f(y+\ell\sin\theta)$ and $f(y-\ell\sin\theta)$ and also take into account the gravity. The force $f=f(s)$ is nonlinear, reflecting the nonlinear elastic behavior of the whole structure, including the action of the sustaining cable. At $s=0$ the elastic force exerted by the hanger balances the gravity, so that the endpoint of the rod is at an equilibrium and $f(0)=0$. Newton's equation for this system reads

$$\frac{m\ell^2}{3}\ddot{\theta}=\ell\cos\theta\big(f(y-\ell\sin\theta)-f(y+\ell\sin\theta)\big),\qquad m\ddot{y}=f(y+\ell\sin\theta)+f(y+\ell\sin\theta)\quad (1)$$

which models a free rod with no interaction with other rods. By analyzing forced and damped versions of system (1), McKenna-Tuama [23,24] were able to numerically show in a cross section a sudden transition from vertical oscillations to torsional oscillations.

We initially consider an isolated system, that is, a model stripped of internal damping and external forcing. If f were linear, then (1) would decouple and there would be no interaction between the y and θ oscillators. We take

$$f(s) = -(s + s^2 + s^3)\ ;$$

this choice satisfies the minimal requirements of being an asymmetric perturbation of a linear force but it is not necessarily expected to yield accurate quantitative information. Numerical experiments show that the qualitative behavior of the system is not affected by the specific choice of the nonlinearity. We then define the positive potential energy $F(s) = -\int_0^s f(\tau)d\tau$.

We label the n rods by $i = 1, \dots, n$: let y_i be the downwards displacement of the center of the i-th rod and θ_i be its angle of deflection from horizontal. Assume that the rods have mass $m = 1$ and half-length $\ell = 1$. Set $y_0 = y_{n+1} = \theta_0 = \theta_{n+1} = 0$ to model the connection between the bridge and the ground. We find the following system of $2n$ equations ($i = 1, \dots, n$):

$$\ddot{\theta}_i + 3U_{\theta_i}(\Theta, \mathrm{Y}) = 0, \qquad \ddot{y}_i + U_{y_i}(\Theta, \mathrm{Y}) = 0, \tag{2}$$

where $(\Theta, \mathrm{Y}) = (\theta_1, \dots, \theta_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}$ and

$$U(\Theta, \mathrm{Y}) = \sum_{i=1}^{n} [F(y_i + \sin\theta_i) + F(y_i - \sin\theta_i)] + \frac{1}{2}\sum_{i=0}^{n} [K_y(y_i - y_{i+1})^2 + K_\theta(\theta_i - \theta_{i+1})^2];$$

here K_y and K_θ are the vertical and torsional stiffness of the bridge. The total energy of the system

$$E\big(\dot{\Theta}, \dot{\mathrm{Y}}, \Theta, \mathrm{Y}\big) = \frac{|\dot{\Theta}|^2}{6} + \frac{|\dot{\mathrm{Y}}|^2}{2} + U(\Theta, \mathrm{Y}) \tag{3}$$

is conserved and is determined by the initial conditions

$$\Big(\dot{\Theta}(0), \mathrm{Y}(0), \Theta(0), \mathrm{Y}(0)\Big) = (\Theta_1, \mathrm{Y}_1, \Theta_0, \mathrm{Y}_0). \tag{4}$$

For all $k = 1, \dots, n$ we compute numerically the initial conditions $\big(\mathrm{Y}_1(k, E_0), \mathrm{Y}_0(k, E_0)\big)$ of the (periodic) k-th nonlinear normal mode of vertical oscillation at energy $E_0 > 0$ and we denote by $T = T(k, E_0)$ its period. The initial condition $(0, \mathrm{Y}_1(k, E_0), 0, \mathrm{Y}_0(k, E_0))$ lies on the orbit of a T-periodic solution to (2) at energy E_0: in this case, one has $\Theta(t) \equiv 0$. We aim to study the torsional stability of the k-th mode, that is, to establish if small Θ-perturbations of such initial condition give rise to wide torsional oscillations. We refer to [5] for a precise definition of nonlinear modes.

We take $n = 16$ and $K_y = K_\theta = 320$. Figure 2 represents the solutions to the system (2) with initial conditions (4) with $|(\Theta_1, \Theta_0)| \ll |(\mathrm{Y}_1, \mathrm{Y}_0)|$ and $(k, E_0) = (1{,}500)$ for $t \in [0{,}200]$. The black plot represents $\theta_i(t)$ whereas the grey plot represents $y_i(t)$, $i = 1, \dots, 8$: we only display these (θ_i, y_i) because $(\theta_i, y_i) \approx (\theta_{17-i}, y_{17-i})$. We also refer to the movies available at the web page [25] where one can see the similarity with the Tacoma collapse and experiments for other modes k. The oscillations of the angular coordinates θ_i are initially small and, suddenly, they become larger.

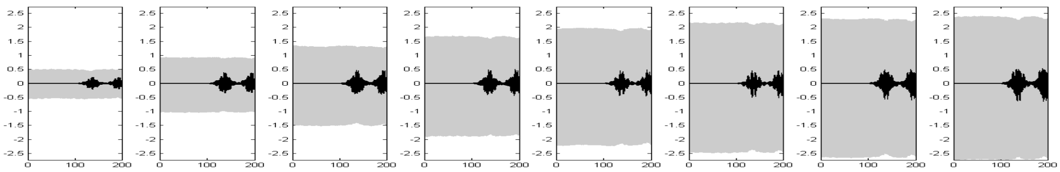

Figure 2. The solution y_i (black) and θ_i (grey) $(i = 1, \dots, 8)$ to (2) with $(k, E_0) = (1{,}500)$.

Numerical evidence suggests that for every k there exists a critical threshold $\bar{E}_k$ for the energy (3): if $E < \bar{E}_k$ then the k-th nonlinear mode of vertical oscillation is stable, while if $E > \bar{E}_k$ it is unstable and a tiny perturbation in any θ_i-variable can lead to wide torsional oscillations.

4 RESONANCES AMONG OSCILLATORS

In order to explain the observed phenomenon we consider now the system (2) with $n = 1$ and $K_y = K_\theta = 1$; we drop the index on θ and y. Here the phase space is 4-dimensional but, due to the conservation of energy, the dynamics takes place in a 3-dimensional manifold. We take a two-dimensional Poincaré section of this manifold. The pictures in Figure 3 represent the $(\dot{\theta}, \theta)$ intersections of some solutions to (1) with the plane $y = 0$ when $\dot{y} > 0$ at energy $E_0 = 3.4$ and $E_0 = 3.8$. The origin is a stable fixed point for the Poincaré map in the picture on the left, whereas it is unstable in the picture on the right. This bifurcation is well-known among specialists of dynamical systems, and it occurs because of a resonance between the oscillators, see [5] for the details. We conclude that, for this problem, the critical energy threshold $\bar{E}$ satisfies $3.4 < \bar{E} < 3.8$.

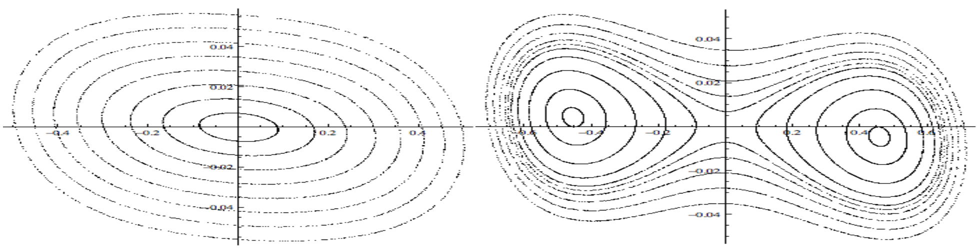

Figure 3. Poincaré maps at energies $E_0 = 3.4$ and $E_0 = 3.8$.

If the maximum modulus of the two eigenvalues of the Jacobian of the Poincaré map at the origin equals 1 the system is stable, whereas if it exceeds 1 the system is unstable. The least energy for which instability is manifested is the threshold $\bar{E}$ for system (2) when $n = 1$.

Back to system (2) with $n > 1$ rods, let $T = T(k, E_0)$ denote the period of the k-th nonlinear normal mode of vertical oscillation at energy $E_0 > 0$ and let $\Psi_{E_0}^k : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be defined by $\Psi_{E_0}^k(\Theta_1, \Theta_0) = \left(\dot{\Theta}(T), \Theta(T)\right)$, where $(\Theta(t), \mathrm{Y}(t))$ solves (2) under the initial conditions (4). The map $\Psi_{E_0}^k$ is an evolution map but it is not a Poincaré map. Still, we evaluate the stability of the k-th mode by computing the eigenvalues of its Jacobian $J\Psi_{E_0}^k(0,0)$ at $(\Theta_1, \Theta_0) = (0,0)$: this is done by solving the equation linearized with respect to Θ at $(\Theta, \mathrm{Y}) = (0, \bar{\mathrm{Y}}_k)$, where $\bar{\mathrm{Y}}_k$ is the k-th nonlinear normal mode at energy E_0, see again [5] for the details. In principle, when $n > 1$ one cannot infer the full stability of the k-th mode from its linear stability which is ensured if all eigenvalues of $J\Psi_{E_0}^{k}(0,0)$ have modulus 1. But we have numerical evidence that the k-th mode is torsionally stable if and only if it is linearly stable. The graphs in Figure 4 display the largest modulus of the eigenvalues of $J\Psi_{E_0}^{k}(0,0)$ as a function of the energy E_0, with $k = 1,2,3$. It appears that for all k there exists $\bar{E}_k$ such that the periodic k-th mode is stable whenever $E < \bar{E}_k$ and it is unstable if E is slightly larger than $\bar{E}_k$. This means that, if $E < \bar{E}_k$ then small initial torsional data yield small torsional behavior for all time, whereas if $E > \bar{E}_k$ then large torsional oscillations suddenly appear also in presence of small initial torsional data. In fact, for higher energies, the system may become stable again, but this has a purely theoretical relevance: in order to ensure that the bridge is torsionally stable its internal energy should be smaller than the least $\bar{E}_k$.

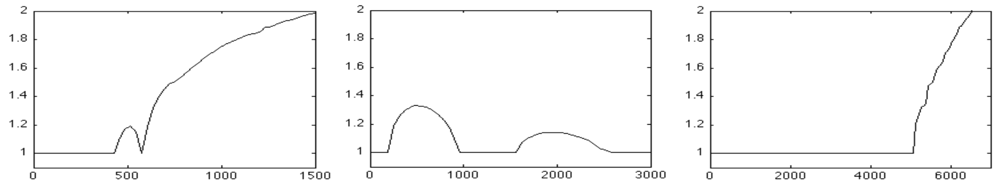

Figure 4. Largest modulus of the eigenvalues of $J\Psi_{E_0}^{k}(0,0)$ versus E_0 for $k = 1,2,3$.

5 APPEARANCE OF TORSIONAL OSCILLATIONS IN DAMPED SYSTEM

The results in this section are new. Let $\delta > 0$ and consider (2) with a damping term:

$$\ddot{\theta}_i + \delta\dot{\theta}_i + 3U_{\theta_i}(\Theta, \mathrm{Y}) = 0, \qquad \ddot{y}_i + \delta\dot{y}_i + U_{y_i}(\Theta, \mathrm{Y}) = 0, \qquad (i = 1, \dots, n) \tag{5}$$

Fix $k \in \{1, \dots, n\}$ and let $\big(\mathrm{Y}_1(k, E_0), \mathrm{Y}_0(k, E_0)\big)$ be as in Section 3. We take as initial conditions

$$\Big(\dot{\Theta}(0), \mathrm{Y}(0), \Theta(0), \mathrm{Y}(0)\Big) = \big(\Theta_1, \mathrm{Y}_1(k, E_0), \Theta_0, \mathrm{Y}_0(k, E_0)\big) \text{ with } 0 < |(\Theta_1, \Theta_0)| < 10^{-6}. \tag{6}$$

We take again $n = 16$. Figure 5 plots the solutions to (5)-(6) for $k = 1$, $\delta = 0.01$ and $\delta = 0.02$, and initial energy $E(0) > 500$, no wide torsion is visible for $E(0) = 500$. These plots should be compared with Figure 2.

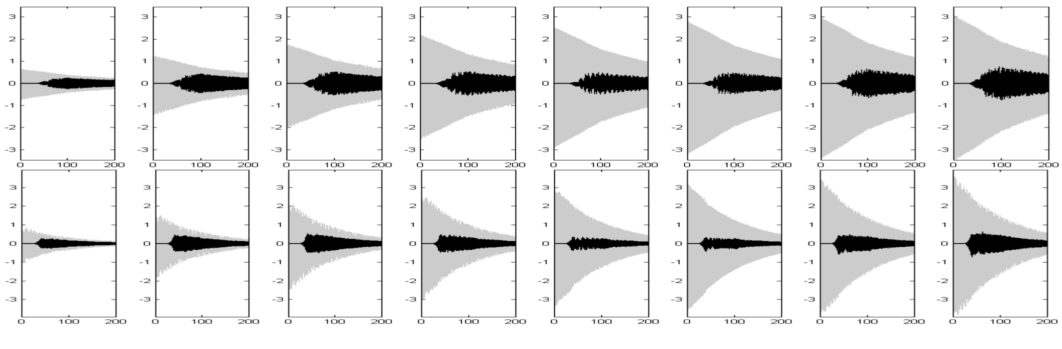

Figure 5: The solution y_i (black) and θ_i (grey) $(i = 1, \dots, 8)$ to (5)-(6) for $(k, E(0)) = (1,1000)$ with $\delta = 0.01$(first line) and $(k, E(0)) = (1,1500)$ with $\delta = 0.02$ (second line), $t \in [0,200]$.

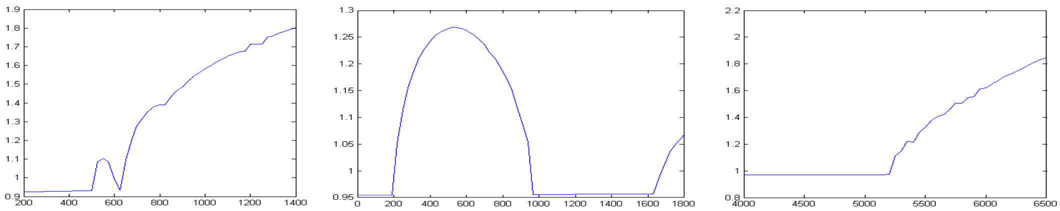

Figure 6. Plot of $\max_l(|\lambda_l^k(E_0)|)$ versus $E(0) = E_0$ for $k = 1,2,3$ and $\delta = 0.1$.

The numerical results for the damped system (5) suggest that for any $\delta > 0$ there exists an energy threshold $\bar{E}_{k,\delta}$ which, if exceeded by the initial energy $E(0) = E_0$ yields torsional instability of the k-th nonlinear normal mode $\overline{\mathrm{Y}}_k(t)$ of (5) at energy E_0, while if $0 < E(0) < \bar{E}_{k,\delta}$ then the system is torsionally stable. Large δ dissipate the energy very rapidly, preventing instability. It appears that the maps $\delta \mapsto \bar{E}_{k,\delta}$ are strictly increasing: if we increase δ from 0 to 0.1, the thresholds $\bar{E}_{k,\delta}$ increase from 450 to 500 ($k = 1$), from 195 to 205 ($k = 2$), from 5100 to 5200 ($k = 3$). To compute $\bar{E}_{k,\delta}$ we use again the evolution map $\Psi_{E_0}^k$ defined above with iteration time equal to the period $T = T(k, E_0)$ of the periodic k-th vertical mode of oscillation at energy $E_0 > 0$. We then compute the $2n$ eigenvalues $\lambda_1^k(E_0), \ldots, \lambda_{2n}^k(E_0)$ of its Jacobian $J\Psi_{E_0}^k(0,0)$ at $(\Theta_1, \Theta_0) = (0,0)$. In turn, the derivatives of $\Psi_{E_0}^k$at $(0,0)$ are computed by linearizing (5) at $(\Theta, \mathrm{Y}) = (0, \overline{\mathrm{Y}}_k)$. Denoting by $\Xi = (\xi_1, \ldots, \xi_n)$ the variation of $\Theta \equiv 0$, this yields the system

$$\ddot{\xi}_i + \delta\dot{\xi}_i + 3\sum_{j=1}^{n} U_{\theta_i\theta_j}(0, \overline{\mathrm{Y}}_k(t))\xi_j = 0 \qquad (i = 1, \ldots, n). \tag{7}$$

The l-th column of $J\Psi_{E_0}^k(0,0)$ is the solution $(\dot{\Xi}(T), \Xi(T))$ at time $T(k, E_0)$of (7) with initial conditions $\left(\dot{\Xi}(0), \Xi(0)\right) = \eta_l$ $(l = 1, \ldots, 2n)$; here η_l is the l-th element of the canonical basis of $\mathbb{R}^{2n}$. It turns out that the solution to (5) with initial data (6) has small $(\dot{\Theta}(t), \Theta(t))$ for all $t > 0$ if and only if $\max_l(|\lambda_l^k(E_0)|) \leq 1$. Therefore, $\bar{E}_{k,\delta} = \inf\left\{E_0 > 0;\ \max_l\left(\left|\lambda_l^k(E_0)\right|\right) > 1\right\}$. Figure 6 should be compared with Figure 4.

6 CONCLUSIONS

An external resonance, the phenomenon which matches the frequency of an external forcing with a natural frequency of the structure, cannot be the reason of the TNB collapse. We have seen why external resonances have to be considered the onset of destructive torsional oscillations. Our model views a suspension bridge as a system of coupled nonlinear oscillators where there can be a sudden transfer of oscillations between the oscillators when large energies are within the structure. Some evolution maps give an effective way to compute the energy threshold where this transfer starts, both in the conservative and damped cases. Any source inserting energy into the structure may overcome the critical energy threshold of the bridge and give rise to uncontrolled oscillations.

In the Report [p.28,4] one finds a letter by Durkee, a project engineer, which states that *There appears to be no difference in the motion whether the wind is steady or gusty*. Hence, the bridge behaves driven by its own internal features, independently of the angle of attack and of the frequency of the wind. Only the amount of energy present in the structure is important. If there is enough energy, then longitudinal oscillations may suddenly switch, without intermediate stages, to destructive torsional oscillations. We believe that the TNB collapsed because that day the critical energy threshold was exceeded by the amount of energy inserted by the wind into the structure. This gave rise to an internal resonance which started the destructive torsional oscillation.

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