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On a Quasilinear Elliptic Differential Equation in Unbounded Domains

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Summary. - *Existence and multiplicity results for a variational quasilinear elliptic equation on unbounded domains are proved; the solutions are obtained as critical points of a nonsmooth functional. We consider the case where the functional is coercive or has a saddle-point geometry.*

1. Introduction

We consider the quasilinear elliptic equation in $\mathbb{R}^n$ $(n \geq 3)$

$$\begin{aligned} -\sum_{i,j} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u = \\ = b(x)u - \lambda u + g(x,u) \ , \end{aligned} \tag{1.1}$$

where the assumptions on a_{ij}, b and g are given in next section, and we determine a weak entire solution in a suitable functional space. To this end, we look for critical points of the functional $J_\lambda : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\begin{aligned} J_\lambda(u) =& \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u) D_i u D_j u - \frac{1}{2}\int_{\mathbb{R}^n} (b(x)-\lambda)u^2 \\ & - \int_{\mathbb{R}^n} G(x,u), \end{aligned} \tag{1.2}$$

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where $G(x,s)=\int_0^s g(x,t)dt$; the main difficulty is that the functional J_λ is not even locally Lipschitz continuous if the functions $a_{ij}(x,s)$ depend on s. However, a more careful analysis of J_λ shows that it has some differentiability properties: as pointed out in [4], the Gâteaux-derivative of J_λ exists at least in the smooth directions; namely, for all $u\in H^1(\mathbb{R}^n)$ and $\phi\in C_c^\infty(\mathbb{R}^n)$ it is possible to evaluate

$$\begin{aligned} J_\lambda'(u)[\phi] &= \int_{\mathbb{R}^n}\sum_{i,j} a_{ij}(x,u)D_iuD_j\phi+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j}\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\phi \\ &- \int_{\mathbb{R}^n}(b(x)-\lambda)u\phi-\int_{\mathbb{R}^n}g(x,u)\phi. \end{aligned}$$

According to the nonsmooth critical point theory developed in [8, 9], the generalized critical points u of J_λ satisfy $J_\lambda'(u)[\phi]=0$ for all $\phi\in C_c^\infty(\mathbb{R}^n)$ and hence solve (1.1) in distributional sense. In Section 3 we briefly recall the basic definitions and properties of this theory and we refer to the original papers for an extensive treatment.

Existence results for (1.1) in a bounded domain were proved in [1]; in this paper we extend these results to $\mathbb{R}^n$: in fact, the statements proved below hold for any unbounded smooth domain.

In (1.1) we assume that $\lambda\geq 0$ and we look for solutions in different functional spaces when $\lambda>0$ or $\lambda=0$; we first prove an existence result for (1.1) in the general case and then a multiplicity result in the case where the functions a_{ij} and g are, respectively, even and odd with respect to u.

2. Statement of the results

We assume an ellipticity condition on the matrix $[a_{ij}(x,s)]$ and a semipositivity condition on the matrix $\left[s\frac{\partial a_{ij}}{\partial s}(x,s)\right]$; more precisely, we assume that there exists $\nu>0$ such that for a.e. $x\in\mathbb{R}^n$, all $s\in\mathbb{R}$ and all $\xi\in\mathbb{R}^n$

$$\sum_{i,j} a_{ij}(x,s)\xi_i\xi_j\geq\nu|\xi|^2 \qquad (2.1)$$

and

$$s \sum_{i,j} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \geq 0. \tag{2.2}$$

We require the coefficients $a_{ij}(x,u)$ and $b(x)$ to satisfy

$$\begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x,s),\ \frac{\partial a_{ij}}{\partial s}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \\ a_{ij}(x,\cdot) \in C^1(\mathbb{R}) \text{ for a.e. } x \in \mathbb{R}^n \\ \lim_{|s|\to\infty} a_{ij}(x,s) = A_{ij}(x) \end{cases} \tag{2.3}$$

and

$$b \in L^{\frac{n}{2}}(\mathbb{R}^n). \tag{2.4}$$

Let $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume that there exist $\alpha \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ and $\beta \in L^{\frac{n}{2}}(\mathbb{R}^n)$ such that

$$|g(x,s)| \leq \alpha(x) + \beta(x)|s| \text{ for all } s \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^n \tag{2.5}$$

and

$$\lim_{|s|\to\infty} \frac{g(x,s)}{s} = 0 \text{ uniformly w.r.t. } x \in \mathbb{R}^n; \tag{2.6}$$

furthermore, if $G(x,s) = \int_0^s g(x,t)dt$, we require that

$$G(x,s) \to +\infty \text{ if } |s| \to \infty \text{ for a.e. } x \in \mathbb{R}^n, \tag{2.7}$$

$$2G(x,s) - sg(x,s) \to +\infty \text{ if } |s| \to \infty \text{ for a.e. } x \in \mathbb{R}^n, \tag{2.8}$$

$$2G(x,s) - sg(x,s) \geq 0 \text{ for a.e. } x \in \mathbb{R}^n \text{ and for all } s \in \mathbb{R} \tag{2.9}$$

and that there exists $\gamma \in L^1(\mathbb{R}^n)$ such that

$$G(x,s) \geq \gamma(x) \text{ for a.e. } x \in \mathbb{R}^n \text{ and for all } s \in \mathbb{R}; \tag{2.10}$$

an example of a function g satisfying the above requirements is given by $g(x,s) = s^{1/3}e^{-|x|}$.

Denote by $\mathcal{D} = \mathcal{D}^{1,2}(\mathbb{R}^n)$ the closure of C_c^∞ (the space of smooth functions in $\mathbb{R}^n$ with compact support) with respect to the norm $\|\phi\|^2 = \int_{\mathbb{R}^n} |\nabla\phi|^2$ and by $H = H^1(\mathbb{R}^n)$ the closure of C_c^∞ with respect to the norm $\|\phi\|_H^2 = \int_{\mathbb{R}^n} |\nabla\phi|^2 + \phi^2$; we consider the standard Hilbert structure on the spaces $\mathcal{D}$ and H. Under the above assumptions we prove:

Theorem 2.1. *Assume (2.1)-(2.10). Then if $\lambda = 0$ equation (1.1) admits a weak solution $u_0 \in \mathcal{D}$, while for all $\lambda > 0$ equation (1.1) admits a weak solution $u_\lambda \in H$.*

In order to establish the geometrical properties of the functional J_0, we consider the linear self-adjoint operator $L^\infty : \mathcal{D} \to \mathcal{D}$ implicitly defined by

$$(L^\infty u, v) = \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i u D_j v - b(x) uv. \tag{2.11}$$

It is well known (see [10] for an extensive treatment of the topic) that under the assumptions we take on A_{ij} and b the whole spectrum $\sigma(L^\infty)$ but a finite set of eigenvalues with finite multiplicity is contained in some interval $[\mu_{\min}, \mu_{\max}]$ with $0 < \mu_{\min} \leq \mu_{\max} < +\infty$. As L^∞ is self-adjoint, there exist orthogonal subspaces $\mathcal{D}^+$, $\mathcal{D}^0$ and $\mathcal{D}^-$ of $\mathcal{D}$ such that $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^-$ and L^∞ is positive definite on $\mathcal{D}^+$, negative definite on $\mathcal{D}^-$ and $\mathcal{D}^0 = \ker L^\infty$; let k be the number of nonpositive eigenvalues, i.e. $k = \dim \mathcal{D}^0 + \dim \mathcal{D}^-$.

Similarly, to consider the case $\lambda > 0$ we define the linear self-adjoint operator $L_\lambda^\infty : H \to H$ by

$$(L_\lambda^\infty u, v)_H = \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i u D_j v - (b(x) - \lambda) uv. \tag{2.12}$$

The spectrum $\sigma(L_\lambda^\infty)$ but a finite set of eigenvalues with finite multiplicity is contained in $[\lambda/2, +\infty)$, the space H splits orthogonally into the positive, null and negative subspaces $H^- \oplus H^0 \oplus H^+$ and if k is the number of nonpositive eigenvalues of L_λ^∞, then $k = \dim H^0 + \dim H^-$.

The equation is said to be resonant when the corresponding linear operator has a nontrivial kernel; the resonant case is in general more difficult to handle because no a priori estimates are available.

If the coefficients of the equation satisfy for all $s \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}^n$ the following symmetries

$$a_{ij}(x,s) = a_{ij}(x,-s) \text{ and } g(x,s) = -g(x,-s), \tag{2.13}$$

then $u \equiv 0$ is a solution of equation (1.1) and nontrivial solutions can be obtained by applying index theory: assume that $\alpha \equiv 0$ in (2.5) and let

$$g_0(x) = \limsup_{s \to 0} \frac{2G(x,s)}{s^2}, \tag{2.14}$$

then $g_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Define the linear self-adjoint operators $L^0 : \mathcal{D} \to \mathcal{D}$ and $L^0_\lambda : H \to H$ by

$$(L^0 u, v) = \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,0) D_i u D_j v - b(x) uv - g_0(x) uv$$

and

$$(L^0_\lambda u, v)_H = \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,0) D_i u D_j v - (b(x) - \lambda) uv - g_0(x) uv.$$

The operators L^0 and L^0_λ have the same properties of L^∞ and L^∞_λ: in particular their positive subspaces have finite codimensions, which we denote by m. We prove the following:

Theorem 2.2. *Assume (2.1)-(2.10) and (2.13). Let $\lambda = 0$ (resp. $\lambda > 0$) and let m and k be defined as above. If $k > m$, then equation (1.1) admits at least $k - m$ pairs of nontrivial weak solutions in $\mathcal{D}$ (resp. H).*

3. Variational setting

We briefly recall some basic definitions of the nonsmooth critical point theory introduced in [8, 9].

Definition 3.1. *Let (X,d) be a metric space, $I \in C(X,\mathbb{R})$ and let $x \in X$. We denote by $|dI|(x)$ the supremum of the $\sigma \in [0,+\infty)$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : B(x,\delta) \times [0,\delta] \longrightarrow B(x,2\delta)$$

such that for all $y \in B(x,\delta)$ and for all $t \in [0,\delta]$ we have

$$d(\mathcal{H}(y,t),y) \leq t \text{ and } I(\mathcal{H}(y,t)) \leq I(y) - \sigma t$$

where $B(x,r) := \{y \in X,\ d(x,y) < r\}$; $|dI|(x)$ is called the weak slope of I at x.

Definition 3.2. *Let (X,d) be a metric space and $I \in C(X,\mathbb{R})$; a point $x \in X$ is said to be critical for I if $|dI|(x) = 0$. A real number c is said to be a critical value for I if there exists $x \in X$ such that $I(x) = c$ and $|dI|(x) = 0$.*

We will prove that the functional J_λ satisfies a weaker version of the Palais-Smale condition which is due to Cerami [6] in the smooth context: in our framework the Palais-Smale-Cerami (PSC) sequences and the PSC condition are defined as follows:

Definition 3.3. *Let X be a Banach space and let $I \in C(X,\mathbb{R})$. A sequence $\{x_m\} \subset X$ is called PSC sequence if $I(x_m)$ is bounded and $(1 + \|x_m\|)|dI|(x_m) \to 0$. We say that I satisfies the PSC condition if all its PSC sequences are precompact.*

Following [1] we introduce

Definition 3.4. *Let X be a Banach space, let $I \in C(X,\mathbb{R})$ and let Y be a dense subspace of X. If the directional derivative of I exists for all x in X in all the directions $y \in Y$ we say that I is weakly Y-differentiable and we call weak Y-slope in x the extended real number*

$$\|I'_Y(x)\|_* := \sup\{I'(x)[\phi] :\ \phi \in Y,\ \|\phi\|_X = 1\}.$$

We can now state the version of the saddle point theorem which we use:

Theorem 3.1. *Let $\lambda = 0$ (resp. $\lambda > 0$), $\mathcal{D} = V \oplus W$ (resp. $H = V \oplus W$), where $V \neq \{0\}$ is finite dimensional; let J_λ be defined as in (1.2) and assume that* *(i)* *J_λ satisfies the PSC condition*

(ii) *there exists $\beta \in \mathbb{R}$ such that $J_\lambda(x) \geq \beta$ for all $x \in W$*

(iii) *there exist $\alpha < \beta$ and $R > 0$ such that $I(x) \leq \alpha$ for all $x \in \partial B_R \bigcap V$*

Then equation (1.1) has a solution $u \in \mathcal{D}$ (resp. $u \in H$) in distributional sense.

Proof. The functional J_λ is of the type

$$J_\lambda(u) = \int_{\mathbb{R}^n} L_\lambda(x, u, \nabla u)dx, \tag{3.1}$$

where $L_\lambda : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions for all $\lambda \geq 0$:

$L_\lambda(x, s, \xi)$ is measurable with respect to x for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$
$L_\lambda(x, s, \xi)$ is of class C^1 with respect to (s, ξ) for a.e. $x \in \mathbb{R}^n$

and there exist $h_1 \in L^1(\mathbb{R}^n)$, $h_2 \in L^1_{loc}(\mathbb{R}^n)$, $h_3 \in L^\infty_{loc}(\mathbb{R}^n)$ and $c \in [0, +\infty)$ such that for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$ the following inequalities hold:

$$\begin{aligned} |L_\lambda(x, s, \xi)| &\leq h_1(x) + c(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\frac{\partial L_\lambda}{\partial s}(x, s, \xi)\right| &\leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\frac{\partial L_\lambda}{\partial \xi}(x, s, \xi)\right| &\leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2); \end{aligned}$$

if $\lambda > 0$, i.e. if we set the problem in the space H, then the first inequality is replaced by the weaker $|L_\lambda(x, s, \xi)| \leq h_1(x) + c(|s|^{\frac{2n}{n-2}} + s^2 + |\xi|^2)$. With the above growth conditions and by adapting Theorem 1.5 in [4] to our case, we infer that J_λ is continuous, weakly $C^\infty_c(\mathbb{R}^n)-$differentiable and that the weak slope gives an upper estimate of the weak $C^\infty_c(\mathbb{R}^n)-$slope, i.e.

$$|dJ_\lambda|(u) \geq \|(J_\lambda)'_{C^\infty_c}(u)\|_*. \tag{3.2}$$

In particular, if u is a critical point of J_λ, then equation (1.1) is satisfied in distributional sense. To complete the proof it suffices to reason as for Theorems 3 and 5 in [1]. $\square$

Remark. If $\lambda > 0$ and $u \in H$ satisfies $|dJ_\lambda|(u) < +\infty$, then it is well known that the conditions (2.1)-(2.5) imply

$$\sum_{i,j} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \in L^1(\mathbb{R}^n)$$

and therefore $J'_\lambda(u)[u]$ is well defined, see [4, 5] for details. The case $\lambda = 0$ and $u \in \mathcal{D}$ can be handled similarly by extending the result in [3] to the following lemma:

Lemma 3.2. *Let Ω be (any) open set in $\mathbb{R}^n$ let $T \in \left[\mathcal{D}^{1,2}(\Omega)\right]^* \cap L^1_{loc}(\Omega)$ and $u \in \mathcal{D}^{1,2}(\Omega)$ satisfying $Tu \geq f$ in Ω for some function $f \in L^1(\Omega)$. Then $Tu \in L^1(\Omega)$ and the duality product $\langle T, u \rangle$ equals $\int_\Omega Tu$.*

Proof. The proof follows by inspection of the proof in [3]. □

If the equation is invariant under a $\mathbb{Z}_2$-action, Theorem 4 in [1] yields:

Theorem 3.3. *Take the same assumptions of Theorem 3.1; assume moreover that J_λ is even and that there exist $\rho, \sigma > 0$ and a subspace U of $\mathcal{D}$ (resp. H) of finite codimension such that*

(iv) $J_\lambda(x) \geq \sigma$ *for all* $x \in \partial B_\rho \bigcap U$

(v) $\operatorname{codim}(U) < \dim(V)$.

Then the equation (1.1) admits at least $\dim(V) - \operatorname{codim}(U)$ *pairs of nontrivial distinct solutions in distributional sense in $\mathcal{D}$ (resp. H).*

4. Preliminary lemmas

Let $\Omega \subset \mathbb{R}^n$ and $p \geq 1$; we set $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{1/p}$ and $\|u\|_p = \left(\int_{\mathbb{R}^n} |u|^p\right)^{1/p}$. The following lemma states that $\int_{\mathbb{R}^n} G(x, u_m)$ is sub-quadratic for diverging u_m.

Lemma 4.1. *Assume (2.5) and (2.6). If $\{u_m\} \subset \mathcal{D}$ (resp. $\{u_m\} \subset H$) is a sequence such that $\|u_m\| \to \infty$ (resp. $\|u_m\|_H \to \infty$), then*

$$\frac{\int_{\mathbb{R}^n} G(x,u_m)}{\|u_m\|^2} \to 0 \quad \left(\text{resp. } \frac{\int_{\mathbb{R}^n} G(x,u_m)}{\|u_m\|_H^2} \to 0 \right) \quad \text{as } m \to \infty.$$

Proof. Let $\{u_m\} \subset \mathcal{D}$ be such that $\|u_m\| \to \infty$; we claim that there exists a sequence $\{\varepsilon_m\} \subset \mathbb{R}^+$ such that $\varepsilon_m \to 0$ and, for a.e. $x \in \mathbb{R}^n$

$$|G(x, u_m(x))| \leq \alpha(x)\|u_m\|^{1/2} + \frac{\beta(x)}{2}\|u_m\| + \varepsilon_m|u_m(x)|^2 \ . \tag{4.1}$$

Take $x \in \mathbb{R}^n$; we prove (4.1) in the case $u_m(x) > 0$, the case $u_m(x) < 0$ being similar. If $u_m(x) < \|u_m\|^{1/2}$ then, by (2.5) we have

$$\begin{aligned} |G(x, u_m(x))| &\leq \int_0^{\|u_m\|^{1/2}} |g(x,t)|dt \leq \int_0^{\|u_m\|^{1/2}} (\alpha(x) + \beta(x) \cdot t)dt \\ &= \alpha(x)\|u_m\|^{1/2} + \frac{\beta(x)}{2}\|u_m\| \end{aligned}$$

and (4.1) follows. If $u_m(x) \geq \|u_m\|^{1/2}$, by Hölder inequality we get

$$\begin{aligned} \int_{\|u_m\|^{\frac{1}{2}}}^{u_m(x)} t\left|\frac{g(x,t)}{t}\right| dt &\leq \left[\int_{\|u_m\|^{\frac{1}{2}}}^{u_m(x)} t^2 dt\right]^{\frac{1}{2}} \cdot \left[\int_{\|u_m\|^{\frac{1}{2}}}^{u_m(x)} \left|\frac{g(x,t)}{t}\right|^2 dt\right]^{\frac{1}{2}} \\ &\leq |u_m(x)|^{\frac{3}{2}} \cdot \varepsilon_m |u_m(x)|^{\frac{1}{2}}, \end{aligned}$$

where ε_m depends on $\|u_m\|$ and by (2.6) $\varepsilon_m \to 0$; combining this with the previous inequality we obtain (4.1).

Choose $\varepsilon > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that $\|\beta\|_{L^{\frac{n}{2}}(\Omega^c)} < \varepsilon$, where $\Omega^c = \mathbb{R}^n \setminus \Omega$. By Hölder inequality, the continuous embedding $\mathcal{D} \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ and (2.5) we have

$$\begin{aligned} &\left|\int_{\Omega^c} G(x, u_m)\right| \leq \\ &\leq \|\alpha\|_{L^{\frac{2n}{n+2}}(\Omega^c)} \|u_m\|_{L^{\frac{2n}{n-2}}(\Omega^c)} + \frac{1}{2}\|\beta\|_{L^{\frac{n}{2}}(\Omega^c)} \|u_m\|^2_{L^{\frac{2n}{n-2}}(\Omega^c)} \\ &\leq c\|u_m\| + \varepsilon c \|u_m\|^2; \end{aligned}$$

furthermore, by integrating (4.1) we have

$$\left|\int_{\Omega} G(x, u_m)\right| \leq c\|u_m\| + \varepsilon_m \|u_m\|^2_{L^2(\Omega)},$$

and these two inequalities yield the result by the arbitrariness of ε. The proof in the H case follows similarly. □

From now on, all the assumptions of Theorem 2.1 are taken. We prove that for every unbounded sequence $\{u_m\}$ such that $J_\lambda(u_m)$ is upper bounded, we can estimate the growth of its norm by means of a suitable local L^2-norm:

Lemma 4.2. *Let $\lambda = 0$ (resp. $\lambda > 0$). There exist a bounded set $\Omega \subset \mathbb{R}^n$ and $\eta > 0$ such that for all sequences $\{u_m\} \subset \mathcal{D}$ (resp. $\{u_m\} \subset H$) satisfying $\sup J_\lambda(u_m) < \infty$ and $\|u_m\| \to \infty$ (resp. $\|u_m\|_H \to \infty$) the following inequality holds:*

$$\|u_m\| \leq \eta \|u_m\|_{L^2(\Omega)} \qquad \left(\textit{resp. } \|u_m\|_H \leq \eta \|u_m\|_{L^2(\Omega)}\right).$$

Proof. We first consider the case $\lambda = 0$ and $\{u_m\} \subset \mathcal{D}$; for all $\varepsilon > 0$ there exist an open bounded set Ω_ε and two functions $b_1 \in L^{\frac{n}{2}}(\mathbb{R}^n)$ and $b_2 \in L^\infty(\mathbb{R}^n)$ such that $b = b_1 + b_2$, $\|b_1\|_{\frac{n}{2}} < \varepsilon$ and $\text{supp} b_2 \subset \Omega_\varepsilon$. Indeed choose Ω_ε so that $\|b\|_{L^{\frac{n}{2}}(\Omega_\varepsilon^c)} < \frac{\varepsilon}{2}$. The restriction of b to Ω_ε is in $L^{\frac{n}{2}}(\Omega_\varepsilon)$, therefore there exist two functions $\tilde{b}_2$ and b_3 such that $\tilde{b}_2 \in L^\infty(\Omega_\varepsilon)$, $b_3 \in L^{\frac{n}{2}}(\Omega_\varepsilon)$ and $\|b_3\|_{L^{\frac{n}{2}}(\Omega_\varepsilon)} < \frac{\varepsilon}{2}$. To conclude take

$$b_1(x) = \begin{cases} b_3(x) & \text{if } x \in \Omega_\varepsilon \\ b(x) & \text{if } x \notin \Omega_\varepsilon \end{cases}$$

and

$$b_2(x) = \begin{cases} \tilde{b}_2(x) & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \notin \Omega_\varepsilon. \end{cases}$$

The result follows by choosing ε small enough, taking into account that $\mathcal{D}$ is continuously embedded into $L^{\frac{2n}{n-2}}$ and setting $\Omega = \Omega_\varepsilon$.

By (2.1) we have

$$J_0(u_m) \geq c\|u_m\|^2 - \int_{\mathbb{R}^n} G(x, u_m) - \frac{1}{2}\int_{\mathbb{R}^n} b(x)u_m^2; \tag{4.2}$$

then, by Lemma 4.1 and the previous observation we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} b(x)u_m^2 \right| &\leq \int_{\mathbb{R}^n} |b_1(x)u_m^2| + \int_{\Omega_\varepsilon} |b_2(x)u_m^2| \\ &\leq \|b_1\|_{\frac{n}{2}} \|u_m^2\|_{\frac{n}{n-2}} + \|b_2\|_\infty \|u_m\|^2_{L^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon \|u_m\|^2 + \|b_2\|_\infty \|u_m\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

In the H case the proof goes similarly: in particular note that inequality (4.2) still holds when changing to the $H-$norm because $\lambda > 0$. $\square$

Lemma 4.3. *All the PSC sequences for J_λ are bounded in $\mathcal{D}$ if $\lambda = 0$ and in H if $\lambda > 0$.*

Proof. We consider the case $\lambda = 0$; the other one follows similarly. By contradiction, let $\{u_m\}$ be a diverging PSC sequence; by the remark in the previous section, for m large we can evaluate $J_0'(u_m)[u_m] - 2J_0(u_m)$ and taking into account (2.2) and (3.2) we have

$$O(1) \geq \int_{\mathbb{R}^n} 2G(x, u_m) - g(x, u_m)u_m. \tag{4.3}$$

Let $v_m(x) := \frac{u_m(x)}{\|u_m\|}$, then there exists $v \in \mathcal{D}$ such that, up to a subsequence, $v_m \rightharpoonup v$ and therefore $v_m \to v$ in L^2_{loc} and $v_m(x) \to v(x)$ for a.e. $x \in \mathbb{R}^n$; Lemma 4.2 implies that $v \neq 0$.

By (2.8) we infer that $2G(x, u_m) - g(x, u_m)u_m \to +\infty$ on a subset of $\mathbb{R}^n$ with positive measure, hence by (2.9) and Fatou Lemma

$$\int_{\mathbb{R}^n} [2G(x, u_m) - g(x, u_m)u_m] \to +\infty,$$

which contradicts (4.3). $\square$

Lemma 4.4. *Let $\{u_m\} \subset \mathcal{D}$ (resp. $\{u_m\} \subset H$) be a PSC sequence for the functional J_0 (resp. J_λ with $\lambda > 0$). Then $\{u_m\}$ is precompact.*

Proof. Let $\{u_m\}$ be a PSC sequence, by Lemma 4.3 $\{u_m\}$ is bounded, hence $u_m \rightharpoonup u$ for some u. By a standard procedure, see e.g. Theorem 2.2.7 in [5], on a subsequence $b(x)u_m \to b(x)u$ and $g(x, u_m) \to g(x, u)$ in $L^{\frac{2n}{n+2}}$; then, by extending to $\mathbb{R}^n$ Theorem 2.1 in [2], by taking into account the local L^2 convergence of $\{u_m\}$ to u and by reasoning as in Lemma 2.3 in [4], we infer that u is a solution in distributional sense of equation (1.1).

If $\lambda = 0$, then the result follows as in [4].

If $\lambda > 0$, by taking the same steps as in the proof of inequality (2.3.10) in [4] we infer that

$$\limsup_{m\to\infty} \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u_m) D_i u_m D_j u_m + \lambda u_m^2 \leq \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u) D_i u D_j u + \lambda u^2;$$

by (2.1) we have

$$\begin{aligned} &\min\{\lambda,\nu\} \|u_m - u\|_H^2 \\ \leq &\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u_m) D_i u_m D_j u_m - 2\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u_m) D_i u_m D_j u \\ &+ \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x,u_m) D_i u D_j u + \lambda \int_{\mathbb{R}^n} \left(u^2 + u_m^2 - 2u_m u\right) \end{aligned}$$

and by Lebesgue dominated convergence theorem we obtain

$$\limsup_{m\to\infty} \|u_m - u\|_H^2 \leq 0$$

which proves that $u_m \to u$ in H. □

5. Proofs of the results

Recall that we defined k to be the number of nonpositive eigenvalues of L^∞ (resp. L_λ^∞) counted with their multiplicity. We first consider the case $\lambda = 0$ and $k \geq 1$ and we prove that the geometrical requirements of the saddle point theorem hold.

Proposition 5.1. *Assume (2.1)-(2.10). Then*

(i) there exists $\beta \in \mathbb{R}$ such that for all $u \in \mathcal{D}^+$ we have $J_0(u) \geq \beta$

(ii) there exist $\alpha < \beta$ and $R > 0$ such that if $u \in \mathcal{D}^- \oplus \mathcal{D}^0$ and $\|u\| = R$, then $J_0(u) \leq \alpha$.

Proof. Since $J_0(u_m)$ is bounded on bounded subsets of $\mathcal{D}$, then (i) holds if $J_0(u_m) \to +\infty$ for every sequence $\{u_m\} \subset \mathcal{D}^+$ such that

$\|u_m\| \to \infty$. Consider a diverging sequence $\{u_m\} \subset \mathcal{D}^+$: by Lemma 4.1 $\int_{\mathbb{R}^n} G(x, u_m)/\|u_m\|^2 \to 0$, therefore it suffices to prove that for m large enough

$$\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m - \int_{\mathbb{R}^n} b(x) v_m^2 \geq c > 0 \ , \qquad (5.1)$$

where $v_m = \frac{u_m}{\|u_m\|}$. There exists $v \in \mathcal{D}$, $\|v\| \leq 1$, such that $v_m \rightharpoonup v$ and $\int_{\mathbb{R}^n} b v_m^2 \to \int_{\mathbb{R}^n} b v^2$ on a subsequence, since $b \in L^{\frac{n}{2}}$.

To prove (5.1) we use the same device as in [7].

Let $l_m = \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m$; as $\{l_m\}$ is bounded, on a subsequence $l_m \to l$ and two cases may occur:

1) $l > \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i v D_j v$. In this case inequality (5.1) follows because $v \in \mathcal{D}^+$.

2) $l \leq \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i v D_j v$. Then by (2.1) we have

$$\begin{aligned} \nu \|v_m - v\|^2 \ &\leq \ \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i (v_m - v) D_j (v_m - v) \\ &= \ \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m \\ &- \ 2 \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v \\ &+ \ \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v D_j v \ ; \end{aligned}$$

but $D_i v_m \rightharpoonup D_i v$ in L^2, and $a_{ij}(x, u_m) D_j v \to A_{ij}(x) D_j v$ in L^2 by Lebesgue dominated convergence theorem, therefore

$$\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v \to \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i v D_j v,$$

$$\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v D_j v \to \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i v D_j v \ ,$$

hence $v_m \to v$ in $\mathcal{D}$ and (5.1) follows.

To prove (ii) it suffices to prove that if $\{u_m\} \subset \mathcal{D}^- \oplus \mathcal{D}^0$ is a diverging sequence, then $J_0(u_m) \to -\infty$. Since $\dim\mathcal{D}^- + \dim\mathcal{D}^0 < +\infty$ and (2.7) holds, then $G(x, u_m) \to +\infty$ on a subset of $\mathbb{R}^n$ with positive measure; by (2.10) and Fatou Lemma we infer

$$\int_{\mathbb{R}^n} G(x, u_m) \to +\infty \ ;$$

the result follows by compactness taking into account (2.2) and the fact that if $u_m \in \mathcal{D}^- \oplus \mathcal{D}^0$, then the quadratic part of the functional is nonpositive. □

Similarly, when $\lambda > 0$ and $k \geq 1$ the following proposition holds:

Proposition 5.2. *Assume (2.3)-(2.10) and let $\lambda > 0$. Then*

(i) there exists $\beta \in \mathbb{R}$ such that for all $u \in H^+$ we have $J_\lambda(u) \geq \beta$.

(ii) there exist $\alpha < \beta$ and $R > 0$ such that if $u \in H^- \oplus H^0$ and $\|u\|_H = R$, then $J_\lambda(u) \leq \alpha$.

Proof. The proof is substantially the same as in Proposition 5.1. We only point out that in order to prove (i), using the same notation of the previous proof, we have to show that for m large

$$\int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m - \int_{\mathbb{R}^n} (b(x) - \lambda) v_m^2 \geq c > 0. \qquad (5.2)$$

Let $l_m = \int_{\mathbb{R}^n} \sum\limits_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m + \lambda \int_{\mathbb{R}^n} v_m^2$; then $l_m \to l$ up to a subsequence.

If $l > \int_{\mathbb{R}^n} \sum\limits_{i,j} A_{ij}(x) D_i v D_j v + \lambda \int_{\mathbb{R}^n} v^2$ we are done.

If $l \le \int_{\mathbb{R}^n} \sum_{i,j} A_{ij}(x) D_i v D_j v + \lambda \int_{\mathbb{R}^n} v^2$, then by (2.1) we have

$$\begin{aligned} \min\{\lambda, \nu\} \|v_m - v\|_H^2 \le & \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v_m \\ & - 2 \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v_m D_j v \\ & + \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, u_m) D_i v D_j v \\ & + \lambda \int_{\mathbb{R}^n} \left(v^2 + v_m^2 - 2 v_m v \right) \end{aligned}$$

and we conclude as in the proof of Proposition 5.1. □

By Lemma 4.4 and the above propositions, the assumptions of Theorem 3.1 are fulfilled and Theorem 2.1 is proved if $k \ge 1$.

If $k = 0$, then L^∞ (resp. L_λ^∞) is positive definite in $\mathcal{D}$ (resp. H), and by the same argument as in the proofs of the previous propositions we infer that J_λ is coercive; furthermore the functional satisfies the PSC condition, therefore it admits a minimum u. By a standard argument of nonsmooth critical point theory [8] we have $|dJ_\lambda|(u) = 0$, hence u is a solution in distributional sense of (1.1) and the proof of Theorem 2.1 is complete.

We prove Theorem 2.2 in the case $\lambda = 0$, the other being similar. By the definition of the operator L^0 there exists a subspace $\mathcal{D}_0^+ \subset \mathcal{D}$ of codimension m such that $(L^0 u, u) \ge c||u||^2$ for all $u \in \mathcal{D}_0^+$. Recall that g_0 was defined in (2.14), then by (2.9) the map $s \longmapsto \frac{G(x,s)}{s^2}$ is not increasing for $s \in [0, +\infty)$, which together with the semipositivity condition (2.2) yields

$$\begin{aligned} J_0(u) &\ge \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x, 0) D_i u D_j u - \frac{1}{2} \int_{\mathbb{R}^n} b(x) u^2 - \frac{1}{2} \int_{\mathbb{R}^n} g_0(x) u^2 \\ &= \frac{1}{2} (L^0 u, u) \end{aligned}$$

for all $u \in \mathcal{D}$; this proves that

$$\liminf_{u\to 0,\ u\in\mathcal{D}_0^+} \frac{J_0(u)}{\|u\|^2} > 0,$$

therefore the hypotheses of Theorem 3.3 are fulfilled and the proof of Theorem 2.2 follows.

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