Long-time behavior of partially damped systems modeling degenerate plates with piers

Filippo GAZZOLA† – Abdelaziz SOUFYANE♯

† Dipartimento di Matematica - Politecnico di Milano, Italy

♯ Department of Mathematics - University of Sharjah, UAE

Abstract

We consider a partially damped nonlinear beam-wave system of evolution PDE's modeling the dynamics of a degenerate plate. The plate can move both vertically and torsionally and, consequently, the solution has two components. We show that the component from the damped beam equation always vanishes asymptotically while the component from the (undamped) wave equation does not. In case of small energies we show that the first component vanishes at exponential rate. Our results highlight that partial damping is not enough to steer the whole solution to rest and that the partially damped system can be less stable than the undamped system. Hence, the model and the behavior of the solution enter in the framework of the so-called *indirect damping* and *destabilization paradox*. These phenomena are valorized by the physical interpretation in the final section, leading to possible new explanations of the Tacoma Narrows Bridge collapse. Several natural problems are left open.

Keywords: degenerate plates, partial damping, longitudinal and torsional components, asymptotic behavior.

AMS 2010 Subject Classification: 35G61, 35B40, 34D05.

Contents

1 Introduction

We consider initial value problems for the following system of PDE's

$$\begin{cases} u_{tt}+\delta u_t+u_{xxxx}+\Big(\displaystyle\int_I(u^2+\theta^2)\Big)u+2\Big(\displaystyle\int_I u\theta\Big)\theta=0 \\ \theta_{tt}-\theta_{xx}+2\Big(\displaystyle\int_I u\theta\Big)u+\Big(\displaystyle\int_I(u^2+\theta^2)\Big)\theta=0 \end{cases} \qquad (x\in I,\, t>0)\,, \tag{1.1}$$

where $I=(-\pi,\pi)$ and $\delta>0$, complemented with the boundary-interior conditions (in which $0<a<1$)

$$u(\pm\pi,t)=u(\pm a\pi,t)=\theta(\pm\pi,t)=\theta(\pm a\pi,t)=0 \qquad t\geq 0. \tag{1.2}$$

Due to the presence of the constraints at $x=\pm a\pi$, solutions of (1.1) fail to be smooth and a suitable notion of weak solution is needed, see Definition 1. The system (1.1) models a degenerate rectangular plate $\Omega=I\times(-\ell,\ell)$ composed by a central beam in position $I\times\{0\}$, whose vertical displacement u is governed by $(1.1)_1$, and by a continuum of cross sections of length $2\ell\ll\pi$ free to rotate around their center placed on the beam, the resulting torsional angle θ being governed by $(1.1)_2$. The plate is divided in three adjacent spans, separated by two fixed piers in the positions $x=\pm a\pi$ that generate the interior constraints in (1.2). Finally, the short edges of the plate at $x=\pm\pi$ are hinged. The view from above of this plate and the position of the piers are drawn in the left picture of Figure 1, in which the beam is

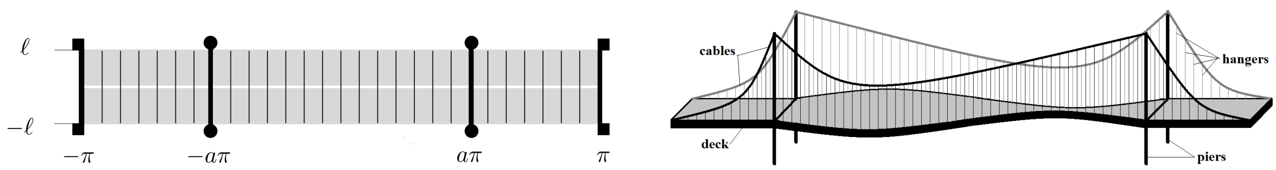

Figure 1: Left: fish-bone model for a degenerate plate with piers. Right: sketch of a suspension bridge.

the white midline and some cross sections are represented by the orthogonal black thin segments. The undamped and unconstrained version of this model was called a *fish-bone* in [13], whereas the constraints due to the piers were introduced in [27, 28] in order to analyze the stability of the deck of suspension bridges, see the right picture in Figure 1 where a torsional displacement is emphasized. The reason of partial damping, acting only in the beam equation $(1.1)_1$, is that while the flexural (longitudinal) displacement u can be damped by stiffening the beam with stronger elastic connections to the ground at $x=\pm\pi$ and to the piers at $x=\pm a\pi$, there is no simple way to damp the torsional displacement θ since the endpoints of the cross sections at $y=\pm\ell$ are free to move. We refer to Section 6 for full details on the physical model and for the explanation why ℓ does not appear explicitly in (1.1).

A few words should be said about the interior constraints in (1.2). On the one hand, the model has its own interest also without these constraints, in which case the plate has no intermediate piers and the solutions of (1.1) are smooth with no need to deal with weak solutions, as in Definition 1. On the other hand, not only the constraints merely require a minor additional effort but, more important, they better describe the behavior of real bridges. They allow to obtain more reliable responses by introducing the precise spatial parameters, see again Section 6. For this reason, the interior constraints appear as an essential part of the model.

The "partially damped" feature of (1.1) raises several natural questions:

does the damping steer the u-component of any solution (u,θ) of (1.1) to zero?
does the damping steer the θ-component of any solution (u,θ) of (1.1) to zero?
if one of the u/θ-components tends to 0, is it possible to determine the decay rate?

Indirect (or partial) damping for coupled systems occurs in the modeling of many real life phenomena such as fluid-structure interaction problems, or dynamical systems from biological sciences, or Timoshenko-beam-systems. The challenging mathematical questions are as above, namely whether one feedback or a minimum number of feedbacks can be regarded as a possible stabilizer for others. The notion of indirect damping mechanisms was introduced by Russell [47] who studied energy dissipation in energy conserving elastic systems with damping coming from coupling with other dissipators. Since then, stabilization results (with the total energy decaying to 0) have been obtained for various kinds of coupled systems including heat-wave, wave-wave, plate-wave, plate-plate, and others. With no hope of being exhaustive, we mention here some works on the stability of coupled linear or nonlinear systems with indirect damping and coupling either within the equations or within the boundary conditions. For fairly different models, in [2, 3, 5, 7, 10, 32, 35, 43, 44, 46, 52, 56, 58] the total energy of the system is shown to decay at different rates, some being integrable at infinity while some others are not. As we shall see in Section 5, this integrability property plays a primary role in the overall behavior of the degenerate plate system (1.1).

In the present paper we show that the response of (1.1) is completely different from all the just mentioned systems. Although the stationary version of (1.1) has a variational structure, and hence a natural energy associated to its solutions, we show that the partial damping is not enough to dissipate the total energy and to steer the whole system to rest (the only stationary solution). We show that only the beam tends to rest and that the rate of decay of its energy seems to depend on the amount of total energy within the system. More precisely, we give the following answers to the above questions:

the u-component of any solution (u, θ) of (1.1) *tends to zero;*
the θ-component of any solution (u, θ) of (1.1) *with $\theta \not\equiv 0$ does not tend to zero;*
for small initial energies, the u-component of any solution (u, θ) of (1.1) *tends to zero exponentially.*

It turns out that, even if the total energy of (1.1) is decreasing, it does not tend to vanish as time goes to infinity and the residual positive energy all concentrates on the torsional component: what happens to the residual energy is an open problem, see Problem 2. Our numerical results in Section 6.3 also show that indirect damping may worsen the stability properties of undamped systems since the θ-component of the undamped system ($\delta = 0$) may be more stable than for the partially damped system (1.1) where $\delta > 0$, see Figures 2-3. The damping parameter conveys part of the initial energy from $(1.1)_1$ to $(1.1)_2$ and, since the torsional displacements are more dangerous for the plate, the system becomes more vulnerable.

The so-called *destabilization paradox* also has a long story, that presumably starts in 1952, when Ziegler [57] observed that the critical force at which a nonconservative finite dimensional system with negligibly low dissipation lost stability was much weaker than that in a system where dissipation was absent from the very beginning. The same paradox has been observed in several mechanical and physical systems [16, 17, 36]. A full understanding of this paradox is still out of reach, in spite of numerous studies on the subject, see [15, 37, 51, 53] and references therein for finite dimensional models; even more obscure appears the infinite dimensional case [38]. The infinite dimensional model (1.1) does not fit in the framework of any of the above studies for at least two reasons. First, the dissipation is not due to gyroscopic forces but it acts directly on only one of the PDE's: this leads to a partial dissipation where the total energy decreases but does not vanish asymptotically. Second, the stability analysis cannot be performed through the linearized system due to the form of the nonlinear coupling terms which leads to a degenerate case with stability eigenvalues having zero real part. Therefore, our stability analysis of (1.1) should be seen as another real-life example of a new form of the destabilization paradox.

The model system (1.1) is fairly simplified and lacks a suitable external forcing term; this will be the subject of future investigation. Nevertheless, even if obtained from an incomplete model, the phenomena highlighted in the present paper have practical applications. Having the Tacoma Narrows Bridge (TNB) collapse [4, 50, 55] in mind, we may afford a new explanation of how longitudinal oscillations switch to torsional oscillations, provoking bridges failures. Placing dampers in the longitudinal direction appears feasible (for instance, by strengthening the central beam), while it is unclear how to reduce possible torsional oscillations. Our results show that

it is not enough to damp the longitudinal oscillations
since they generate torsional oscillations which persist for all time.

And since the torsional oscillations were considered the main cause of the TNB collapse, see Section 6.1, this suggests that

the TNB collapse was induced by the longitudinal damping
and damping in future bridges should be placed with great care.

The proofs of the results are obtained by combining energy bounds with several delicate estimates for some additional auxiliary functionals associated to (1.1). Our analysis of the system (1.1) starts by showing that the total energy associated to (1.1) is decreasing along its solutions. The particular form of (1.1) enables us to partially decouple the total energy and, by introducing two auxiliary functionals (denoted by h and F in the proof of Theorem 1), we obtain some differential inequalities showing that the part of the energy relative to u tends to vanish at infinity. Yet, this says nothing about the rate of decay nor about the behavior of the θ-component. Therefore, we introduce a functional containing the nonlinear part of (1.1) (denoted by a in Section 4) and, through a change of unknown function, we show that if it is sufficiently small, then the u-component vanishes exponentially fast, see Theorem 2. In order to ensure that the nonlinearity is small, we assume that the initial energy is small. If this condition fails, we are not able to determine the rate of decay of u and also affording a conjecture appears as a challenge, see Problem 1. These results are then used to prove that any solution of (1.1) (with $\theta \not\equiv 0$) has nonvanishing θ-component. The proof of this fact, stated in Theorem 3, uses the quadratic integrability of u to show that the θ-energy remains bounded away from 0 so that, as a classic property of oscillators, θ_x cannot die off at infinity. We complement this result with some numerics for a particular bimodal system, see (6.10), which highlights the increment of the amplitude of the torsional oscillation occurring for both synchronized and asynchronized initial data, although the latter lead to a smaller increment. This is why we believe that a complete theoretical proof is quite challenging, see Problem 2.

This paper is organized as follows. In Section 2 we set up the functional framework by defining the phase space and what is meant by weak solution. We also recall some spectral properties of the linear differential operators involved. In Section 3 we state and prove that the u-component of any solution of (1.1) tends to vanish as time goes to infinity. In Section 4 we state and prove that the u-component vanishes exponentially fast if the initial energy is sufficiently small. In Section 5 we state and prove that all the nontrivial solutions of (1.1) have a nonvanishing torsional component. The new phenomena highlighted in the present paper are valorized in Section 6 where we briefly recall the behavior of bridges, we quote the numerical results, and we give physical interpretations to all our results.

2 The phase space and weak solutions

We introduce the spaces

$$V(I) := \{u \in H^2 \cap H^1_0(I);\, u(\pm a\pi) = 0\}\,, \qquad W(I) := \{u \in H^1_0(I);\, u(\pm a\pi) = 0\} \tag{2.1}$$

and we notice that the boundary and internal conditions

$$u(-\pi) = u(\pi) = u(-a\pi) = u(a\pi) = 0 \tag{2.2}$$

are well defined since $V(I) \subset W(I) \subset C^0(\overline{I})$. These spaces are Hilbert spaces when endowed with the scalar products

$$(u, v)_V = \int_I u''v''\,, \qquad (u, v)_W = \int_I u'v'\,.$$

We denote, respectively, by $V'(I)$ and $W'(I)$ the dual spaces of $V(I)$ and $W(I)$ and by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ the duality products; the space $L^2(I)$ is the pivot space and the related norm will be denoted by $\|\cdot\|_2$. It is shown in [28] that $V(I)$ is a subspace of $H^2\cap H^1_0(I)$ having codimension 2, whose orthogonal complement is made by piecewise third order polynomials which do not match C^3 at $\pm a\pi$. In fact, $V(I)^\perp$ contains functions that are more regular than $H^2(I)$; they are $C^2(\overline{I})$, but they fail to be C^3 (except for the zero function) since each pier produces a discontinuity in the third derivative.

We then consider the set of the eigenvalues μ and the corresponding eigenfunctions $e\in V(I)$ solving the problem

$$\int_I e''v''=\mu\int_I ev \qquad \forall v\in V(I). \tag{2.3}$$

In our context, we also need to consider the following second order eigenvalue problem in $W(I)$:

$$\int_I e'w'=\nu\int_I ew \qquad \forall w\in W(I). \tag{2.4}$$

Denoting by $\mu_1=\lambda_1^4$ the least eigenvalue of (2.3) and by $\nu_1=\kappa_1^2$ the least eigenvalue of (2.4), we have the two Poincaré-type inequalities

$$\lambda_1^4\|v\|_2^2\le\|v_{xx}\|_2^2 \quad \forall v\in V(I)\,, \qquad \kappa_1^2\|w\|_2^2\le\|w_x\|_2^2 \quad \forall w\in W(I)\,. \tag{2.5}$$

We then denote

$$\Lambda=\min\{\lambda_1^4,\kappa_1^2\}\,. \tag{2.6}$$

Numerical results in [28] seem to show that the minimum is κ_1^2, regardless of the position of the piers. We point out that the operator L defined on $V(I)$ by $\langle Lu,v\rangle_V=\int_I u''v''$ *is not* the square of the operator $\mathcal{L}$ defined on $W(I)$ by $\langle\mathcal{L}u,v\rangle_W=\int_I u'v'$. Moreover,

$$\text{the eigenfunctions of (2.3) and (2.4) are qualitatively different} \tag{2.7}$$

since the latter are identically zero on at least one of the spans and, thereby, nonsmooth.

Thanks to the spaces $V(I)$ and $W(I)$, introduced in (2.1), we may define weak solutions of (1.1).

Definition 1. *We say that the functions*

$$u\in C^0(\mathbb{R}_+;V(I))\cap C^1(\mathbb{R}_+;L^2(I))\cap C^2(\mathbb{R}_+;V'(I))$$
$$\theta\in C^0(\mathbb{R}_+;W(I))\cap C^1(\mathbb{R}_+;L^2(I))\cap C^2(\mathbb{R}_+;W'(I))$$

are weak solutions of (1.1)-(1.2) *if*

$$\langle u_{tt},\varphi\rangle_V+\delta\int_I u_t\varphi+\int_I u_{xx}\varphi''+\int_I(u^2+\theta^2)\cdot\int_I u\varphi+2\int_I u\theta\cdot\int_I\theta\varphi=0, \tag{2.8}$$

$$\langle\theta_{tt},\psi\rangle_W+\int_I\theta_x\psi'+2\int_I u\theta\cdot\int_I u\psi+\int_I(u^2+\theta^2)\cdot\int_I\theta\psi=0, \tag{2.9}$$

for all $(\varphi,\psi)\in V(I)\times W(I)$ *and all* $t>0$*, where the spaces* $V(I)$ *and* $W(I)$ *are defined in* (2.1).

We emphasize that the junction conditions (1.2) are "hidden" in the property that $u(t)\in V(I)$ and $\theta(t)\in W(I)$ for all $t\ge0$. We complement (2.8)-(2.9) with some initial conditions

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad \theta(x,0)=\theta_0(x),\quad \theta_t(x,0)=\theta_1(x) \qquad x\in I, \tag{2.10}$$

assuming that $u_0\in V(I)$, $\theta_0\in W(I)$, $u_1,\theta_1\in L^2(I)$. We recall the well-posedness of this problem, as obtained in [27, Proposition 4.1].

Proposition 1. *For all $u_0 \in V(I)$, $\theta_0 \in W(I)$, $u_1, \theta_1 \in L^2(I)$, there exists a unique weak solution (u, θ) of* (2.8)-(2.9)-(2.10)*. Moreover, $u \in C^2(\overline{I} \times \mathbb{R}_+)$ and $u_{xx}(-\pi, t) = u_{xx}(\pi, t) = 0$ for all $t > 0$.*

Solutions $(u(t), \theta(t))$ of (2.8) belong to the phase space $V(I) \times W(I)$ for all t:

we call u the longitudinal component of the solution and θ the torsional component of the solution.

If $\theta_0(x) = \theta_1(x) = 0$ (resp., $u_0(x) = u_1(x) = 0$) in (2.10), then the solution of (2.8)-(2.9)-(2.10) satisfies $\theta(x, t) \equiv 0$ (resp., $u(x, t) \equiv 0$). These initial conditions for θ or u give rise to purely longitudinal (resp., purely torsional) solutions. But, as we shall see, the interesting dynamics in (2.8)-(2.9) occurs whenever both the longitudinal and torsional components are nonzero, creating a coupling with possible energy transfer between the two components.

3 Vanishing of the longitudinal component

In this section we prove that the longitudinal component of any solution of (2.8)-(2.9) tends to vanish.

Theorem 1. *For all $u_0 \in V(I)$, $\theta_0 \in W(I)$, $u_1, \theta_1 \in L^2(I)$, the longitudinal component of the corresponding solution (u, θ) of* (2.8)-(2.9)-(2.10) *satisfies*

$$\lim_{t\to\infty} \left(\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2\right) = 0\,.$$

Proof. To the system (2.8)-(2.9) we associate the energy

$$E(t) = \frac{\|u_t(t)\|_2^2}{2} + \frac{\|\theta_t(t)\|_2^2}{2} + \frac{\|u_{xx}(t)\|_2^2}{2} + \frac{\|\theta_x(t)\|_2^2}{2} + \frac{\|u(t)+\theta(t)\|_2^4}{8} + \frac{\|u(t)-\theta(t)\|_2^4}{8}\,, \tag{3.1}$$

which turns out to be a crucial tool in the analysis of the behavior of (2.8)-(2.9). Note that the "quartic term" (which is reminiscent of von Kármán theory [54], see [30]) can also be written as

$$\frac{\|u(t)+\theta(t)\|_2^4}{8} + \frac{\|u(t)-\theta(t)\|_2^4}{8} = \frac{\left(\|u(t)\|_2^2 + \|\theta(t)\|_2^2\right)^2}{4} + \left(\int_I u(t)\theta(t)\right)^2 .$$

By using (2.8)-(2.9) we readily see that

$$\dot{E}(t) = -\delta \|u_t(t)\|_2^2 \tag{3.2}$$

so that $t \mapsto E(t)$ is nonincreasing and, in particular,

$$\exists E_\infty := \lim_{t\to\infty} E(t) \in \big[0, E(0)\big]\,, \qquad E_\infty \le E(t) \le E(0) \quad \forall t \ge 0\,. \tag{3.3}$$

Moreover, the identity

$$E(0) \ge E(0) - E_\infty = -\int_0^\infty \dot{E}(t)\,dt = \delta \int_0^\infty \|u_t(t)\|_2^2\,dt \tag{3.4}$$

shows that $t \mapsto \|u_t(t)\|_2^2$ is integrable over $(0, \infty)$.

Notice that Definition 1 merely ensures that $u_t \in C^0(\mathbb{R}_+; L^2(I))$ and, apparently, one cannot take $\varphi = u_t(t)$ in (2.8) in order to obtain (3.2). However, the choice $\varphi = u_t(t)$ is allowed by noticing that (2.8)-(2.9) are satisfied in the sense of distributions in the three intervals I_-, I_0, I_+ (although not on the whole interval I), which allows to split any integral over I as the sum of three integrals; moreover, u and θ are smooth classical solutions on each of these intervals, which allows to argue as in [33, Theorem 5.2.1]. In fact, one can also invoke the finite-dimensional Galerkin approximation, for which solutions are smooth and any test is allowed, and then take the limit.

Then we define the "u-energy"

$$F(t) := \frac{\|u_t(t)\|_2^2}{2} + \frac{\|u_{xx}(t)\|_2^2}{2} + \frac{\|u(t)\|_2^4}{4}\,.$$

By taking $\varphi = u_t(t)$ in (2.8) we get

$$\dot F(t) = -\|\theta(t)\|_2^2 \int_I u(t)u_t(t) - 2\int_I u(t)\theta(t)\cdot\int_I \theta(t)u_t(t) - \delta\|u_t(t)\|_2^2\,. \tag{3.5}$$

Next, we consider the function

$$h(t) = \frac{\|u(t)\|_2^2}{2}$$

so that $\dot h(t) = \int_I u(t)u_t(t)$ and, by using (2.8) with $\varphi = u(t)$,

$$\begin{aligned}\ddot h(t) &= \langle u_{tt}(t), u(t)\rangle_V + \|u_t(t)\|_2^2\\ &= -\delta\int_I u(t)u_t(t) - \|u_{xx}(t)\|_2^2 - \|u(t)\|_2^4 - \|\theta(t)\|_2^2\|u(t)\|_2^2 - 2\left[\int_I \theta(t)u(t)\right]^2 + \|u_t(t)\|_2^2\,.\end{aligned}$$

By recalling the above expression of $\dot h(t)$, we may rewrite this identity as

$$\|u_{xx}(t)\|_2^2 = -\ddot h(t) - \delta\dot h(t) - \|u(t)\|_2^4 - \|\theta(t)\|_2^2\|u(t)\|_2^2 - 2\left[\int_I \theta(t)u(t)\right]^2 + \|u_t(t)\|_2^2\,.$$

Integrating over $[0,T]$ for some $T>0$ and dropping the negative terms, we obtain

$$\int_0^T \|u_{xx}(t)\|_2^2 dt \le \int_0^T \|u_t(t)\|_2^2 dt + \dot h(0) - \dot h(T) + \delta h(0)\,.$$

Then we notice that

$$|\dot h(T)| \le \int_I |u(T)u_t(T)| \le \frac{\|u(T)\|_2^2 + \|u_t(T)\|_2^2}{2} \le C\,E(0)$$

in view of (2.5) and (3.3). By letting $T\to\infty$ and recalling (3.4), we then obtain

$$\int_0^\infty \|u_{xx}(t)\|_2^2 dt < \infty\,.$$

Therefore, also $t\mapsto \|u(t)\|_2^2$ and $t\mapsto\|u(t)\|_2^4$ are integrable over $(0,\infty)$ and, hence,

$$F \in L^1(0,\infty). \tag{3.6}$$

In particular, this proves that there exists an increasing sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = +\infty\,, \qquad \lim_{n\to\infty} F(t_n) = 0 \tag{3.7}$$

since, otherwise, the integral $\int_0^\infty F$ would diverge. From now on, we denote by ω_n positive numbers, that may vary from line to line and also within the same line, such that

$$\lim_{n\to\infty}\omega_n = 0$$

in which the index n is the same as in (3.7). In particular, from $F\in L^1(0,\infty)$ and (3.7) we infer that

$$F(t_n) = \omega_n \quad\text{and}\quad \int_{t_n}^\infty F(s)\,ds = \omega_n\,. \tag{3.8}$$

Finally, we notice that, by (3.5),

$$\dot{F}(t) \le KF(t) \qquad \text{for some } K > 0\,.$$

Fix an integer n and integrate this inequality over $[t_n, t]$ for $t > t_n$ to obtain

$$F(t) \le F(t_n) + K\int_{t_n}^{t} F(s)\,ds = \omega_n \qquad \forall t > t_n$$

where the second equality follows from (3.8). This is none other than the definition of limit, that is, this shows that

$$\lim_{t\to\infty} F(t) = 0\,.$$

This completes the proof of Theorem 1. □

Theorem 1 leaves open two questions. First, the rate of decay of the longitudinal component, that we address in Section 4. Second, Theorem 1 nothing says about the behavior of the torsional component: we analyze this problem in Section 5.

4 Exponential decay of longitudinal components for small energies

In this section we show that if the initial data are sufficiently small, then the longitudinal component of the solution of (2.8)-(2.9) decays exponentially.

Theorem 2. *Let $E = E(t)$ be as in (3.1). There exists $\beta > 0$ such that if $u_0 \in V(I)$, $\theta_0 \in W(I)$, $u_1, \theta_1 \in L^2(I)$ are small enough in such a way that*

$$E(0) \le \beta\,, \tag{4.1}$$

then there exists $C, \eta > 0$ such that the longitudinal component of the corresponding solution (u, θ) of (2.8)-(2.9)-(2.10) *satisfies*

$$\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 \le Ce^{-\eta t} \qquad \forall t \ge 0\ .$$

Proof. Let (u, θ) be a weak solution of (2.8)-(2.9) (according to Definition 1) and set $a(t) := \|u(t)\|_2^2 + \|\theta(t)\|_2^2$ so that, by (4.1) and (3.3),

$$\begin{aligned}
\beta \quad &\ge \quad \frac{\|u_{xx}(t)\|_2^2}{2} + \frac{\|\theta_x(t)\|_2^2}{2} + \frac{\|u(t)+\theta(t)\|_2^4}{8} + \frac{\|u(t)-\theta(t)\|_2^4}{8} \\
\text{by (2.5)} \quad &\ge \quad \frac{\lambda_1^4}{2}\|u(t)\|_2^2 + \frac{\kappa_1^2}{2}\|\theta(t)\|_2^2 + \frac{1}{4}\Big(\|u(t)\|_2^2 + \|\theta(t)\|_2^2\Big)^2 \\
\text{by (2.6)} \quad &\ge \quad \frac{\Lambda}{2}a(t) + \frac{1}{4}a(t)^2
\end{aligned}$$

from which we infer that

$$a(t) < \frac{2\beta}{\Lambda} \qquad \forall t \ge 0\,. \tag{4.2}$$

We prove Theorem 2 by fixing σ such that

$$0 < \sigma < \frac{\delta}{2}\,, \qquad 25\sigma(\delta-\sigma)^2 \le 4\lambda_1^4(\delta - 2\sigma) \tag{4.3}$$

and, next, taking

$$\beta = \Lambda\sigma(\delta-\sigma)\,. \tag{4.4}$$

With these choices, we observe that

$$-\sigma\delta+\sigma^2 \le a(t)-\sigma\delta+\sigma^2 \le \sigma\delta-\sigma^2 \implies \big|a(t)-\sigma\delta+\sigma^2\big| \le \sigma\delta-\sigma^2 \qquad \forall t\ge 0\,. \tag{4.5}$$

Then, we rewrite (2.8) as

$$\langle u_{tt},\varphi\rangle_V+\delta\int_I u_t\varphi+\int_I u_{xx}\varphi''+a(t)\int_I u\varphi+2\int_I \theta u\cdot\int_I\theta\varphi=0$$

which, if we put $w(t):=u(t)e^{\sigma t}$, becomes

$$\langle w_{tt},\varphi\rangle_V+\int_I w_{xx}\varphi''=-(\delta-2\sigma)\int_I w_t\varphi+\sigma(\delta-\sigma)\int_I w\varphi-2\int_I\theta w\cdot\int_I\theta\varphi-a(t)\int_I w\varphi$$

for all $t\in[0,T]$ and all $\varphi\in V(I)$. Formally take $\varphi=w_t(t)$ so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\Big] &= \big[\sigma\delta-\sigma^2-a(t)\big]\int_I w(t)w_t(t)-(\delta-2\sigma)\|w_t(t)\|_2^2-2\int_I\theta(t)w(t)\int_I\theta(t)w_t(t)\\ &\le \Big(2a(t)+\big|a(t)+\sigma^2-\sigma\delta\big|\Big)\|w(t)\|_2\|w_t(t)\|_2-(\delta-2\sigma)\|w_t(t)\|_2^2\,,\end{aligned}$$

where we used the Hölder inequality and the fact that $\|\theta(t)\|_2^2\le a(t)$. We then use the first Poincaré inequality in (2.5) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\Big] &\le \frac{2a(t)+\big|a(t)+\sigma^2-\sigma\delta\big|}{\lambda_1^2}\|w_{xx}(t)\|_2\|w_t(t)\|_2-(\delta-2\sigma)\|w_t(t)\|_2^2\\ &\le \gamma\|w_{xx}(t)\|_2\|w_t(t)\|_2-(\delta-2\sigma)\|w_t(t)\|_2^2\end{aligned}$$

where, by (4.2)-(4.4)-(4.5),

$$\gamma<\frac{4\beta/\Lambda+\sigma\delta-\sigma^2}{\lambda_1^2}=\frac{5\sigma(\delta-\sigma)}{\lambda_1^2}\,. \tag{4.6}$$

By combining this with the Young inequality

$$\|w_{xx}(t)\|_2\|w_t(t)\|_2\le\frac{1}{2}\,\frac{\gamma}{\sqrt{(\delta-2\sigma)^2+\gamma^2}+\delta-2\sigma}\|w_{xx}(t)\|_2^2+\frac{\sqrt{(\delta-2\sigma)^2+\gamma^2}+\delta-2\sigma}{2\gamma}\|w_t(t)\|_2^2\,,$$

we get

$$\frac{d}{dt}\Big[\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\Big]-\alpha\Big[\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\Big]\le 0 \qquad \text{with } \alpha=\frac{\gamma^2}{\sqrt{(\delta-2\sigma)^2+\gamma^2}+\delta-2\sigma}\,.$$

Hence,

$$\frac{d}{dt}\Big[e^{-\alpha t}\big(\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\big)\Big]\le 0$$

and, upon integration over $(0,t)$, we finally infer

$$\|w_t(t)\|_2^2+\|w_{xx}(t)\|_2^2\le\big(\|w_t(0)\|_2^2+\|w_{xx}(0)\|_2^2\big)e^{\alpha t} \qquad \forall t\ge 0\,.$$

By undoing the change of unknowns and going back to u, we get

$$\|u_t(t)+\sigma u(t)\|_2^2+\|u_{xx}(t)\|_2^2\le\big(\|u_1+\sigma u_0\|_2^2+\|(u_0)_{xx}\|_2^2\big)e^{(\alpha-2\sigma)t} \qquad \forall t\ge 0\,.$$

Hence, Theorem 2 is proved if we show that

$$\eta:=2\sigma-\alpha>0\,. \tag{4.7}$$

By recalling the expression of α, we see that (4.7) is equivalent to

$$2\sigma(\delta-2\sigma)+2\sigma\sqrt{(\delta-2\sigma)^2+\gamma^2}>\gamma^2\,.$$

In order to prove this fact, we notice that

$$2\sigma(\delta-2\sigma)+2\sigma\sqrt{(\delta-2\sigma)^2+\gamma^2}>4\sigma(\delta-2\sigma)\geq\frac{25\sigma^2(\delta-\sigma)^2}{\lambda_1^4}>\gamma^2\,,$$

where the first inequality is obvious, the second inequality is a consequence of (4.3), the third inequality is a consequence of (4.6). Hence, also the claimed inequality (4.7) is proved and the exponential decay of u follows. ☐

Remark 1. Recently, for a fully damped evolution equation and with a fairly different proof, Haraux [34, Corollary 4.3] also obtained the exponential decay of solutions with small initial energy.

Problem 1. Theorem 2 leads to a natural question: does the longitudinal component of a solution of (1.1) vanish exponentially also in presence of large initial energies? For the answer, we have different hints leading to opposite feelings. In favor of a positive answer, we studied some bimodal solutions of (1.1) (of the form (6.9) below) and we numerically saw that $t\mapsto e^{\delta t/2}w(t)$ has a finite (positive) limsup for small initial data: this suggests exponential decay with the same rate as the linear equation $\ddot{w}+\delta\dot{w}+\gamma w=0$ for $\gamma>\delta^2/4$. By increasing the initial energy, we then detected a slower decay, still at exponential rate but with exponent smaller than $\delta/2$. Moreover, the plot of the graph of the solution of the problem $\ddot{y}+2\dot{y}+(3+\cos 2t)y+y^3=0$ with $y(0)=1$ and $\dot{y}(0)=0$ suggests polynomial decay. For these reasons, in our opinion, a full answer to this question is extremely challenging.

5 Nonvanishing of the torsional component

In this section we show that all the solutions of (2.8)-(2.9) with initial data $(\theta_0,\theta_1)\neq(0,0)$ have a nonvanishing torsional component.

Theorem 3. *Assume that* $(\theta_0,\theta_1)\neq(0,0)$. *Then the torsional component* $\theta=\theta(t)$ *of the corresponding solution* (u,θ) *of* (2.8)-(2.9)-(2.10) *satisfies*

$$\limsup_{t\to\infty}\|\theta_x(t)\|_2>0\,. \tag{5.1}$$

Proof. Assume for contradiction that (5.1) does not hold, then

$$\lim_{t\to\infty}\|\theta_x(t)\|_2=0 \tag{5.2}$$

and, from (2.9) and Theorem 1, also

$$\lim_{t\to\infty}\theta_{tt}(t)=0\quad\text{in } W'(I)\,. \tag{5.3}$$

By taking $\psi=\theta(t)$ in (2.9), we see that

$$\begin{aligned}\frac{d}{dt}\int_I\theta(t)\theta_t(t) &= \|\theta_t(t)\|_2^2+\langle\theta_{tt}(t),\theta(t)\rangle_W\\ &= \|\theta_t(t)\|_2^2-\|\theta_x(t)\|_2^2-2\left(\int_I u(t)\theta(t)\right)^2-\int_I u(t)^2\cdot\int_I\theta(t)^2-\left(\int_I\theta(t)^2\right)^2.\end{aligned}$$

Upon integration, this shows that, for all $t > 0$,

$$\begin{aligned}\int_t^{t+1} \|\theta_t(s)\|_2^2\, ds &= \int_I \theta(t+1)\theta_t(t+1) - \int_I \theta(t)\theta_t(t) \\ &+ \int_t^{t+1} \left[\|\theta_x(s)\|_2^2 + 2\left(\int_I u(s)\theta(s)\right)^2 + \int_I u(s)^2 \int_I \theta(s)^2 + \left(\int_I \theta(s)^2\right)^2 \right] ds\end{aligned}$$

and all the terms in the r.h.s. tend to zero as $t \to +\infty$ in view of Theorem 1 and the assumption (5.2): also recall that all the terms are bounded due to (3.3). Together with (5.3) this shows that also

$$\lim_{t\to\infty} \|\theta_t(t)\|_2 = 0 \,. \tag{5.4}$$

Hence, if we define the "θ-energy" functional

$$G(t) := \frac{\|\theta_t(t)\|_2^2}{2} + \frac{\|\theta_x(t)\|_2^2}{2} + \frac{\|\theta(t)\|_2^4}{4} \,, \tag{5.5}$$

from (5.2) and (5.4) we deduce

$$\lim_{t\to\infty} G(t) = 0 \,. \tag{5.6}$$

Moreover, by (2.9), we infer that

$$\dot{G}(t) = -2 \int_I u(t)\theta(t) \cdot \int_I u(t)\theta_t(t) - \int_I u(t)^2 \cdot \int_I \theta(t)\theta_t(t) \tag{5.7}$$

and, by repeated use of the Hölder inequality, we obtain (for some $c > 0$)

$$-\dot{G}(t) \le 3\|u(t)\|_2^2 \|\theta(t)\|_2 \|\theta_t(t)\|_2 \le c\varepsilon(t)G(t) \qquad \text{with } \varepsilon(t) := \|u(t)\|_2^2 \,.$$

After division by $G(t) > 0$ and upon integration over $(0, t)$, the last inequality yields

$$G(0) \le e^{c\int_0^t \varepsilon(s)ds} G(t) \,.$$

By (3.6) we infer that $\varepsilon \in L^1(0, \infty)$ so that, by letting $t \to \infty$ and by using (5.6), the last inequality implies that $G(0) = 0$, which contradicts the assumption that $(\theta_0, \theta_1) \neq (0, 0)$. This contradiction completes the proof. □

Problem 2. As shown by Theorem 1, the longitudinal component of (2.8)-(2.9) tends to vanish as $t \to \infty$. Moreover, the total energy of the system is decreasing in view of (3.2). On the other hand, (5.1) shows that some "residual energy" remains in the system, all concentrated on the torsional component. One then wonders what kind of shape has the "residual torsional oscillation". For bimodal systems such as (6.10) below, it is natural to conjecture that θ tends to become periodic with a period depending on the amount of residual energy which, however, appears difficult to quantify. Even harder seems to be the case of multi-modal torsional components, since the residual energy may be spread on different θ-modes with the possible appearance of almost periodic solutions.

6 Physical interpretation of the results in a suspension bridge

6.1 Some historical facts

Several bridges suffered unexpected oscillations both during construction and after inauguration, sometimes also leading to collapses, see e.g. [1]. Thanks to the videos available on the web [55], most people have seen the spectacular collapse of the Tacoma Narrows Bridge (TNB), occurred in 1940: the torsional oscillations were considered the main cause of this dramatic event [4, 50]. But torsional oscillations leading to failures also appeared in several other bridges. Let us just mention the collapses of the Brighton Chain Pier (1836), of the Menai Straits Bridge (1839), of the Wheeling Suspension Bridge (1854), of the Matukituki Suspension Footbridge (1977). We refer to [29, Chapter I] for more historical events.

Here we just recall some of the phenomena that can be explained through our results by focusing on the specific case of the TNB. On the day of the collapse, the bridge was oscillating longitudinally as may other times before. According to Eldridge [4, V-3], a witness of the TNB collapse, *the bridge appeared to be behaving in the customary manner ... oscillating in a four noded manner ... these motions, however, were considerably less than had occurred many times before.* Nevertheless, a sudden change in the motion was alarming, a violent destructive torsional movement started. A witness to the collapse was Farquharson, the man escaping in the video [55]: according to his detailed testimony in [23], *a violent change in the motion was noted. This change appeared to take place without any intermediate stages and with such extreme violence that the span appeared to be about to roll completely over.* Farquharson then continues with a fundamental description of the modes: *The motion, which a moment before had involved a number of waves (nine or ten) had shifted almost instantly to two.* This transition to torsional oscillations was unexpected [4, p.31]: *Prior to 10:00 A.M. on the day of the failure, there were no recorded instances of the oscillations being otherwise than the two cables in phase and with no torsional motions.* Nowadays everybody agrees *the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*, see [50, p.63].

Since 1940, may attempts of explanations were made by engineers, physicists, mathematicians and... others. In an article published in the New York Times [42] a few days after the collapse, one reads *Like all suspension bridges, that at Tacoma both heaved and swayed with a high wind. It takes only a tap to start a pendulum swinging. Time successive taps correctly and soon the pendulum swings with its maximum amplitude. So with the bridge. What physicists call resonance was established, with the result that the swaying and heaving exceeded the limits of safety.* Clearly, who wrote these lines was not a scientist. The mathematicians Lazer-McKenna [39, § 1] remark that *the phenomenon of linear resonance is very precise. Could it really be that such precise conditions existed in the middle of the Tacoma Narrows, in an extremely powerful storm*? The physicists Green-Unruh [31] mention that *making the comparison to a forced harmonic oscillator requires that the wind generates a periodic force tuned to the natural frequency of the bridge.* The engineers, Billah-Scanlan [14] make a fool of physics textbooks who attempt to explain the TNB collapse with an aerodynamic resonance. Hence, mechanical resonance, intended as a perfect matching between the exterior wind and the parameters of the bridge, is not the culprit for the TNB collapse. Billah-Scanlan [14, p.121] also claim that Scanlan-Tomko [49] *...demonstrated conclusively that the catastrophic mode of the old Tacoma Narrows bridge was a case of what they termed single-degree-of-freedom torsional flutter due to complex, separated flow.* McKenna [41] wrote that Billah-Scanlan [14] *...offered a mathematical model which is only valid for very small displacements and can only be verified in ideal wind tunnel experiments of "in torsion* $0 \leq \alpha \leq \pm 3^o$*". We are asked to believe that these "penetrating insights" explain the Tacoma Narrows oscillation. To us, the case is less than convincing.* With some sarcasm McKenna comments by writing that *apparently the authors were not familiar with the concept of absolute value* and he concludes by saying that [14] *...is a perfectly good explanation of something that was never observed, namely small torsional oscillations, and no explanation of what did occur, namely a large vertical oscillation with a double amplitude of five ft. and a frequency of 38 per min. followed by a change to the torsional.* And indeed, from the Report [4, p.31] we recall that torsional oscillations were never recorded prior to the day of the TNB collapse. Finally, McKenna concludes that *...if the explanation in [14] has any validity, why were small torsional oscillations never observed? After all, the bridge was known to have oscillated vertically in winds of 3 m.p.h., and remained motionless in winds of 35 m.p.h., (when according to [14], "divergent amplitudes" are reached). It is also worth noting that the bridge had survived winds of 48 m.p.h. without undergoing torsional oscillations, [4], page 28.*

Not much progress was made since then. In 1978, Scanlan [48, p.209] writes *The original Tacoma Narrows Bridge withstood random buffeting for some hours with relatively little harm until some fortuitous condition "broke" the bridge action over into its low antisymmetrical torsion flutter mode.* In 1999, McKenna [40, § 2.3] writes that *there is no consensus on what caused the sudden change to torsional motion.* In 2001, Scott [50] writes *Opinion on the exact cause of the Tacoma Narrows Bridge collapse is even today not unanimously shared.* So, there seems to be no convincing explanation why torsional oscillations appear: of course, a "fortuitous condition" is not an explanation. This open question is the main motivation for the present paper.

A collection of observed phenomena shows that the mode of oscillation, even more than the amplitude, is responsible for the switch to torsional oscillations. From the Report [4, p.20] we learn that, in the months prior to the collapse, *one principal mode of oscillation prevailed and the modes of oscillation frequently changed.* Moreover, oscillations with more than 10 nodes on the three spans were never seen [4, p.28]. This is why, for some nonlinear evolution beam equations, the notion of prevailing mode was introduced in [26]. And this is also the reason why, at least for qualitative results, we may consider a (finite) system of ODE's instead of the original PDE system (1.1). The two-modes system (6.10), numerically analyzed in the next subsection, precisely considers the ninth longitudinal mode coupled with the second torsional mode, whose frequencies are determined by (2.3) and (2.4).

It is clear that in absence of wind or external loads the deck of a bridge remains still. When the wind hits a the deck of a bridge the flow creates vortices which generate a forcing lift that starts the longitudinal oscillations of the deck. This explanation is accepted by the entire scientific community and it has been studied with great precision in wind tunnel tests, see e.g. [50]. This is the point where our analysis starts, namely when the longitudinal oscillations of the bridge reach an apparently periodic motion which is maintained in amplitude by a somehow perfect equilibrium between the input of energy from the wind and internal dissipation. Then, we "switch off the wind" (of course, this is possible in wind tunnels) so that the structure is only subject to internal (longitudinal) dissipation, thereby obtaining (1.1). In this situation, we have seen that the longitudinal oscillations tend to vanish (Theorems 1 and 2), while the torsional oscillations persist in time (Theorem 3). In the next subsection we numerically study these phenomena in a simplified system with the parameters of the collapsed TNB.

6.2 The physical model

The system (1.1) (and its weak form (2.8)-(2.9)) was suggested in [27] as a model for suspension bridges. The parameter $0 < a < 1$ determines the relative measure of the side spans with respect to the main span and most suspension bridges have equal side spans with

$$\frac{1}{2} \leq a \leq \frac{2}{3}.$$

The presence of destructive torsional oscillations in bridges (see Section 6.1) suggests to rule out beam models and to consider instead plate models [12, 18, 24, 25, 29, 30]. If one digits "tacoma narrows bridge collapse images" on Google, one finds pictures of the wide oscillations prior to the TNB collapse and one sees that, while a torsional motion is visible on the main span, the side spans do not display torsional displacements. From a mathematical point of view, this means that

the matching between the displacements on the three spans is in general not smooth.

This fact is confirmed by a careful look at the video [55] which clearly shows that, during the oscillations, the connection between the main span and the side spans *is not* C^1. In fact, the pictures on the web and the video also show that

within the deck, only the displacement of the midline is smooth during the oscillations.

This is why among several possible ways of modeling the deck of a bridge with piers, we chose a degenerate plate, composed by a beam representing the midline of the plate and by cross sections that are free to rotate around the beam, see Figure 1. This guarantees both a smooth midline displacement and nonsmooth connections between spans. We consider symmetric side spans so that, after setting

$$I = (-\pi, \pi), \quad I_- = (-\pi, -a\pi), \quad I_0 = (-a\pi, a\pi), \quad I_+ = (a\pi, \pi),$$

the plate is identified with the planar rectangle

$$\Omega = I \times (-\ell, \ell) \subset \mathbb{R}^2,$$

while the three spans are identified with

$$\Omega_0 := I_0 \times (-\ell, \ell) \quad \text{(main span)}, \qquad \Omega_- = I_- \times (-\ell, \ell), \quad \Omega_+ = I_+ \times (-\ell, \ell) \quad \text{(side spans)}.$$

The white midline in Figure 1 divides the roadway into two lanes and its vertical displacement is denoted by $u = u(x,t)$, for $x \in I$ and $t > 0$. The equilibrium position of the midline is $u = 0$, with

$u > 0$ *corresponding to a downwards displacement.*

Each cross-section is free to rotate around the midline and its angle of rotation is denoted by $\alpha = \alpha(x,t)$. The vertical displacements of the two endpoints of the cross sections (in position x and at time t) are given by

$$u(x,t) + \ell \sin \alpha(x,t) \qquad \text{and} \qquad u(x,t) - \ell \sin \alpha(x,t). \tag{6.1}$$

Since we are not interested in describing accurately the behavior of the plate under large torsional angles, for small α the following approximations are legitimate:

$$\cos \alpha \cong 1 \quad \text{and} \quad \sin \alpha \cong \alpha. \tag{6.2}$$

If we set $\theta = \ell\alpha$, this cancels the dependence on the width ℓ, see [13] for the details. In view of (6.2), the displacements (6.1) now read $u(x,t) \pm \theta(x,t)$.

We derive the Euler-Lagrange equations for this structure using variational methods, as a consequence of an energy balance. Denoting by $M > 0$ the mass density, the kinetic energy of the central beam and of the rod having half-length ℓ are (respectively) given by

$$\frac{M}{2}\int_I u_t^2\,, \qquad \frac{M}{6}\ell^2 \int_I \alpha_t^2 = \frac{M}{6}\int_I \theta_t^2. \tag{6.3}$$

Moreover, there exists a constant $\mu > 0$, depending on the shear modulus and on the moment of inertia of the pure torsion, such that the total potential energy of the cross sections is given by

$$\frac{\mu\ell^2}{2}\int_I \alpha_x^2 = \frac{\mu}{2}\int_I \theta_x^2. \tag{6.4}$$

The bending energy of the beam depends on its curvature: if $EI > 0$ is the flexural rigidity of the beam, it is given by

$$\frac{EI}{2}\int_I u_{xx}^2.$$

Finally, the most delicate energy terms, which create the *coupling* between the longitudinal displacement u and the torsional angle θ, are generated by the potentials

$$G(u+\theta) = \frac{\gamma}{4}\left(\int_I (u+\theta)^2\right)^2, \qquad G(u-\theta) = \frac{\gamma}{4}\left(\int_I (u-\theta)^2\right)^2. \tag{6.5}$$

This potential, considered in [19] for a single beam, is a good measure for the geometric nonlinearity due to the displacements. In (6.5), not only the constant $\gamma > 0$ measures the nonlinearity of the system but it also measures the strength of the coupling between u and θ: in the limit case $\gamma = 0$ the overall system is linear and uncoupled. A cubic nonlinearity naturally arises when large deflections of a beam or a plate are involved: in this case, the stretching effects suggest to use variants of the von Kármán theory [54], see also [20, 21] for a modern point of view and [30] for the adaptation of this theory to plates modeling bridges. In fact, when dealing with bridges, the nonlinearity should as well take into account the behavior of the sustaining cables and, for this reason, also the engineering literature deals with cubic nonlinearities, see e.g. [6, 8, 45]. By taking this into account, several cubic nonlinearities were considered in [27] and the most reliable (better describing the dynamics of a bridge) appeared to be the nonlocal one appearing in (1.1), although different nonlinearities may also be considered [11].

By putting all the above terms together and by taking all the constants (except γ) equal to 1, we find that the total energy of the system (2.8)-(2.9) is given by

$$\mathcal{E}(u,\theta) = \frac{1}{2}\int_I (u_t^2 + \theta_t^2 + u_{xx}^2 + \theta_x^2) + \frac{\gamma}{4}\left[\int_I (u+\theta)^2\right]^2 + \frac{\gamma}{4}\left[\int_I (u-\theta)^2\right]^2 . \tag{6.6}$$

If we take $\gamma = 1/2$ and we emphasize its dependence on time, we obtain the energy in (3.1). This energy balance yields the equations (1.1) that should be intended in the weak form (2.8)-(2.9).

Problem 3. In order to derive (1.1) we assumed small displacements, see (6.2). Clearly, this is allowed if the interest is focused on small torsional oscillations, which is the case if one merely seeks the instability of purely longitudinal oscillations. But, as we shall see below, torsional oscillations may grow up quickly and, therefore, the linearization is no longer legitimate. If we do not assume (6.2), then we do not use (6.3) and (6.4) while (6.5) becomes

$$G(u+\ell\sin\alpha) = \frac{\gamma}{4}\left(\int_I (u+\ell\sin\alpha)^2\right)^2, \qquad G(u-\ell\sin\alpha) = \frac{\gamma}{4}\left(\int_I (u-\ell\sin\alpha)^2\right)^2 .$$

Does the resulting system exhibit different behaviors than (2.8)-(2.9)?

6.3 Bimodal solutions and numerical results

We now interpret the results stated throughout this paper and we complement them with some numerics. We aim to reproduce some of the phenomena described in Section 6.1. In particular, we wish to emphasize the possible change of oscillations from longitudinal to torsional.

In order to evaluate as well the role of the nonlinearity, we maintain the parameter γ in (6.6) thereby obtaining a slightly more general system than (2.8)-(2.9):

$$\begin{gathered}\langle u_{tt},\varphi\rangle_V + \delta\int_I u_t\varphi + \int_I u_{xx}\varphi'' + 2\gamma\int_I (u^2+\theta^2)\cdot\int_I u\varphi + 4\gamma\int_I u\theta\cdot\int_I\theta\varphi = 0,\\ \langle\theta_{tt},\psi\rangle_W + \int_I\theta_x\psi' + 4\gamma\int_I u\theta\cdot\int_I u\psi + 2\gamma\int_I(u^2+\theta^2)\cdot\int_I\theta\psi = 0,\end{gathered} \tag{6.7}$$

for all $(\varphi,\psi) \in V(I)\times W(I)$ and all $t > 0$; obviously, (6.7) coincides with (2.8)-(2.9) when $\gamma = 1/2$.

Let e_λ be an L^2-normalized eigenfunction of (2.3) related to the eigenvalue λ^4 and let η_κ be an L^2-normalized eigenfunction of (2.4) related to the eigenvalue κ^2. There is a natural coupling between these modes and, for some of them, the presence of the piers yields an additional coupling measured by the coefficient

$$A_{\lambda,\kappa} = A_{\lambda,\kappa}(a) := \int_I e_\lambda\eta_\kappa .$$

Note that $A_{\lambda,\kappa}^2 < 1$ in view of the Hölder inequality: recall that $e_\lambda \not\equiv \eta_\kappa$, see (2.7). Moreover, if e_λ and η_κ have opposite parities, then $A_{\lambda,\kappa} = 0$. But there are also cases where $A_{\lambda,\kappa} \neq 0$, see [28]. If $A_{\lambda,\kappa} = 0$, then the space $\langle e_\lambda\rangle \times \langle \eta_\kappa\rangle$ is invariant. This means that if, for some real numbers c_1, c_2, c_3, c_4, we take initial data such as

$$(u_0, u_1) = (c_1, c_2)e_\lambda\,, \qquad (\theta_0, \theta_1) = (c_3, c_4)\eta_\kappa\,, \tag{6.8}$$

then the solution of (6.7) has the form

$$\Big(u(x,t), \theta(x,t)\Big) = \Big(w(t)e_\lambda(x), z(t)\eta_\kappa(x)\Big)\,. \tag{6.9}$$

We call solutions such as (6.9) *bimodal solutions.*

For the TNB one had that $a = 14/25$ and, in such case, the ninth longitudinal eigenvalue is $\lambda^4 \approx 633$ whereas the second torsional eigenvalue is $\kappa^2 \approx 3.189$, see [28]. These eigenvalues correspond to the oscillations seen the day of the collapse, as described in Section 6.1. The associated eigenfunctions e_λ and η_κ have opposite parities so that, according to what we have just explained, we have $A_{\lambda,\kappa} = 0$. Then we take initial data as (6.8) so that the solution of (6.7) has the form (6.9). By plugging (6.9) into (6.7) we see that the couple (w, z) solves the system

$$\begin{cases} \ddot{w}(t) + \delta\dot{w}(t) + 633\, w(t) + 2\gamma\big(w(t)^2 + z(t)^2\big)w(t) = 0 \\ \ddot{z}(t) + 3.189\, z(t) + 2\gamma\big(w(t)^2 + z(t)^2\big)z(t) = 0 \end{cases} \qquad (t \geq 0)\,. \tag{6.10}$$

From Theorem 2 we know that $w(t) \to 0$ as $t \to \infty$ (vanishing longitudinal component) while from Theorem 3 we know that $z(t) \not\to 0$ (nonvanishing torsional component). We numerically studied system (6.10) with initial conditions (6.8) and the most surprising result was the following.

Main numerical result (I): partial damping increases torsional oscillations.

★ With $\gamma = 25$, $c_1 = 5$, $c_3 = 0.01$, $c_2 = c_4 = 0$, $\delta = 0.1$, we obtained the plots in Figure 2 where an increment of the amplitude of the torsional oscillation is quite visible: it goes from 0.01 (initially) to more than 0.03, yielding a rate of increment for z of more than a factor 3.

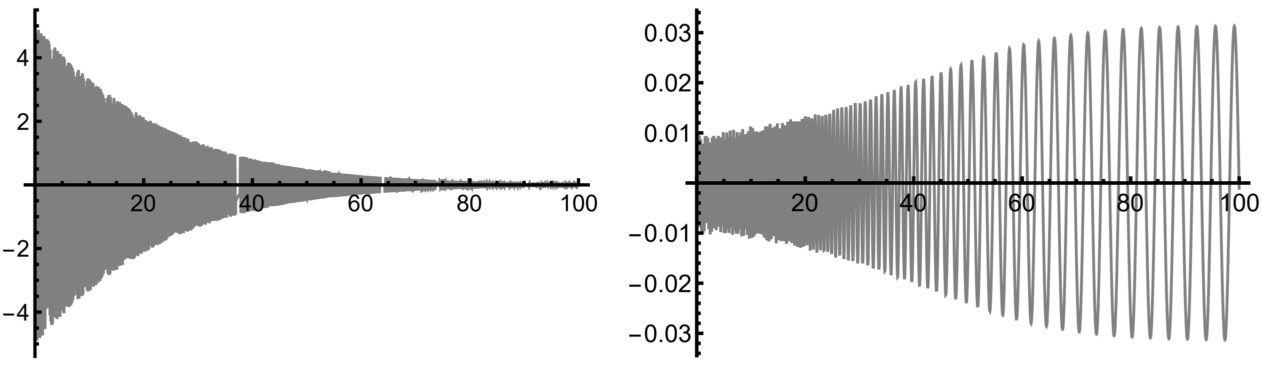

Figure 2: For $t \in [0, 100]$, plot of the longitudinal component w (left) and of the torsional component z (right) of the solution of (6.10) with $c_1 = 5$, $c_3 = 0.01$, $c_2 = c_4 = 0$ ($\delta = 0.1$).

★ With the very same parameters, except for the damping parameter that was set to be $\delta = 0$ (no damping), we obtained the plots in Figure 3 that should be compared with Figure 2. It is evident that in Figure 3 the longitudinal component maintains the same amplitude of oscillation (as expected since the first equation is undamped) and that the torsional component *does not* increment its amplitude. The explanation is that, with this choice of the parameters, the system (6.10) is of the kind

$$\ddot{w}(t) + \lambda^4 w(t) + 2\gamma\big(w(t)^2 + z(t)^2\big)w(t) = 0\,, \quad \ddot{z}(t) + \kappa^2 z(t) + 2\gamma\big(w(t)^2 + z(t)^2\big)z(t) = 0$$

with $\lambda^4 > \kappa^2$. Then we know from [27, Proposition 4.2] that the longitudinal mode is linearly stable with respect to the torsional mode which means that, regardless of the initial data, one does not

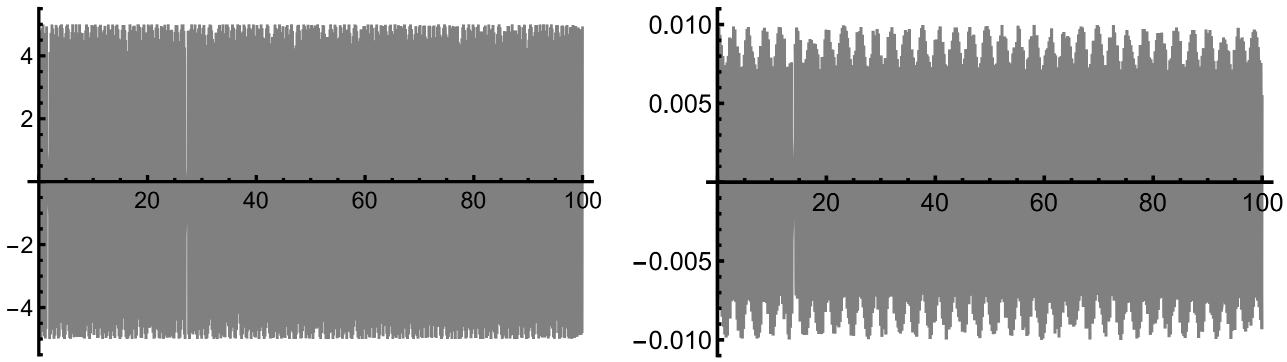

Figure 3: For $t \in [0, 100]$, plot of the longitudinal component w (left) and of the torsional component z (right) of the solution of (6.10) with $c_1 = 5$, $c_3 = 0.01$, $c_2 = c_4 = 0$ ($\delta = 0$).

expect an increment of the torsional oscillations. On the contrary, Figure 2 shows that the torsional oscillations may increase whenever $\delta > 0$. This enables us to reach the unexpected conclusion stated in the Introduction, namely

although the longitudinal damping decreases the total energy, see (3.2),
it may increase the (destructive) torsional energy.

In other words, partial damping can lead to disasters! □

Main numerical result (II): the role of synchronization.
For system (6.10) with $\gamma = 25$, the torsional energy (5.5) and its derivative (5.7) become

$$G(t) = \frac{\dot z(t)^2}{2} + 3.189\,\frac{z(t)^2}{2} + \frac{25}{2}\,z(t)^4\,, \qquad \dot G(t) = -50w(t)^2 z(t)\dot z(t)\,.$$

Therefore, for the previous experiment where $c_1 = 5$, $c_3 = 0.01$, $c_2 = c_4 = 0$, we initially have that $\dot G(t) > 0$, as long as $z(t)\dot z(t) < 0$, see Figure 4 (left) where we depict the graph of $t \mapsto G(t)$ for $t \in [0, 0.2]$. In order to understand if the increment of torsional oscillations (see the right picture in Figure 2) is due this initial increment of G, we tried some asynchronized initial data. For the same $\delta = 0.1$, we chose $c_1 = 5$, $c_4 = 0.0178$, $c_2 = c_3 = 0$, namely initial data yielding the very same initial energy but, now, with $\dot G(t) < 0$ on a first interval of time, as long as $z(t)\dot z(t) > 0$, see Figure 4 (right).

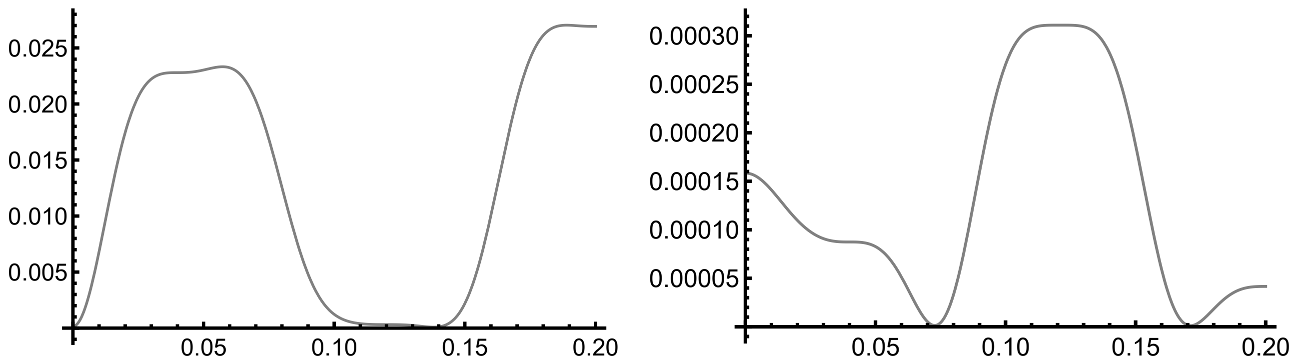

Figure 4: For $t \in [0, 0.2]$, plots of $t \mapsto G(t)$ for the two above choices of initial conditions.

The resulting solution of (6.10) is depicted in Figure 5, to be compared with Figure 2. It turns out

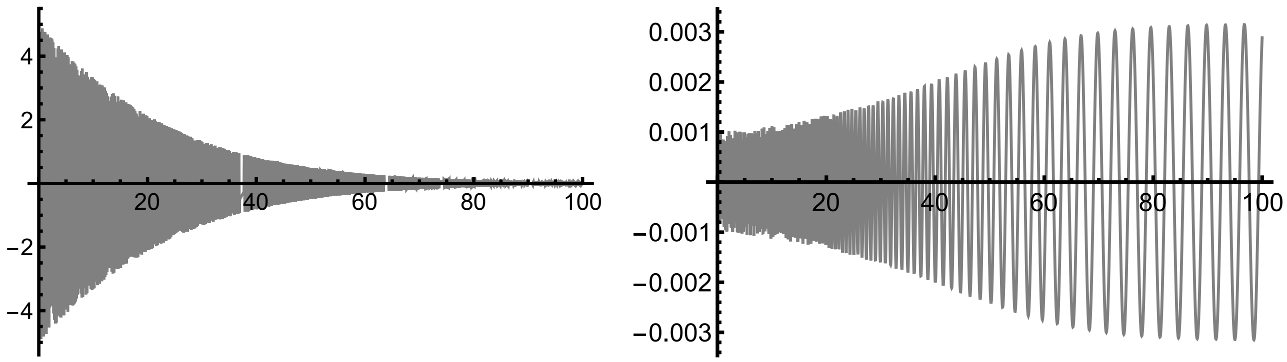

Figure 5: For $t \in [0, 100]$, plot of the longitudinal component w (left) and of the torsional component z (right) of the solution of (6.10) with $c_1 = 5$, $c_4 = 0.0178$, $c_2 = c_3 = 0$ ($\delta = 0.1$).

that the longitudinal part w has qualitatively the same behavior while the torsional part, although increasing in amplitude, increases much less than in Figure 2. This suggests that

any choice of the initial data leads to an increment of the torsional energy, although asynchronized data lead to a smaller increment.

This means that asynchronization can only mitigate the energy transfer but it cannot eliminate it! As a consequence, it appears out of reach to prove a general statement on the energy transfer, regardless of the initial conditions. □

Further numerical results: variation of the involved parameters.

• By increasing δ we saw that the transfer of energy occurred more quickly: this means that stronger dampers anticipate the appearance of torsional oscillations. Quite surprisingly, variations of δ do not affect the rate of increment of z. Hence, the only difference seems to occur between $\delta = 0$ and $\delta > 0$.

• By increasing γ we saw that the rate of increment of z was larger: it tends to 1 as $\gamma \to 0$ and it tends to some (apparently) finite limit as $\gamma \to \infty$, always increasing.

• The rate of increment of z was decreasing with respect to c_3, reaching a positive limit value as $c_2 \to 0$, reaching 1 when $c_3 \approx c_1$, becoming less than 1 for $c_3 > c_1$. This means that if the torsional oscillation is initially large, then it is also damped. Overall, the parameter influencing the rate of growth was the ratio c_1/c_3.

• By plotting the function $t \mapsto e^{\delta t/2} w(t)$ we a found a finite nonzero limit as $t \to \infty$ for small initial data. This seems to say that the longitudinal component decays exponentially to 0 like $e^{-\delta t/2}$, as in the linear case. For large initial data the decay was still exponential but at a lower rate. We could not detect a precise rule nor decays other than exponential.

• We also tested the response of the system for varying a (position of the piers) but we could not detect a simple rule governing this variation. This topic certainly deserves more attention and will also be the object of subsequent investigation. □

Acknowledgements. The Authors are grateful to an anonymous Referee and to the Editors who suggested to insert the topic of the present paper into the framework of the destabilization paradox.

The first Author is partially supported by the PRIN project *Direct and inverse problems for partial differential equations: theoretical aspects and applications* and by the GNAMPA group of the INdAM.

References

[1] B. Akesson, *Understanding bridges collapses*, CRC Press, Taylor & Francis Group, London (2008)

[2] F. Alabau, P. Cannarsa, V. Komornik, *Indirect internal stabilization of weakly coupled evolution equations*, J. Evol. Equ. 2, 127-150 (2002)

[3] F. Alabau-Boussouira, Z. Wang and L. Yu, *One-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities*, ESAIM COCV 23, 721–749 (2017)

[4] O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941)

[5] F. Ammar-Khodja, S. Kerbal, A. Soufyane, *Stabilization of the nonuniform Timoshenko beam*, J. Math. Anal. Appl. 327, 525-538 (2007)

[6] G. Augusti, V. Sepe, *A "deformable section" model for the dynamics of suspension bridges. Part I: Model and linear response*, Wind and Structures 4, 1-18 (2001)

[7] G. Avalos, I. Lasiecka, R. Triggiani, *Heat-wave interaction in 2-3 dimensions: optimal rational decay rate*, J. Math. Anal. Appl. 437, 782-815 (2016)

[8] G. Bartoli, P. Spinelli, *The stochastic differential calculus for the determination of structural response under wind*, J. Wind Engineering and Industrial Aerodynamics 48, 175-188 (1993)

[9] U. Battisti, E. Berchio, A. Ferrero, F. Gazzola, *Periodic solutions and energy transfer between modes in a nonlinear beam equation*, J. Math. Pures Appl. 108, 885-917 (2017)

[10] C. Batty, L. Paunonen, D. Seifert, *Optimal energy decay for the wave-heat system on a rectangular domain*, SIAM J. Math. Anal. 51, 808-819 (2019)

[11] V. Benci, D. Fortunato, F. Gazzola, *Existence of torsional solitons in a beam model of suspension bridge*, Arch. Ration. Mech. Anal. 226, 559-585 (2017)

[12] E. Berchio, A. Ferrero, F. Gazzola, *Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions*, Nonlin. Anal. Real World Appl. 28, 91-125 (2016)

[13] E. Berchio, F. Gazzola, *A qualitative explanation of the origin of torsional instability in suspension bridges*, Nonlin. Anal. TMA 121, 54-72 (2015)

[14] K.Y. Billah, R.H. Scanlan, *Resonance, Tacoma Narrows Bridge failure, and undergraduate physics textbooks*, Amer. J. Physics 59, 118-124 (1991)

[15] A. Bloch, P.S. Krishnaprasad, J.E. Marsden, *Dissipation induced instabilities*, Annales Inst. H. Poincaré Anal. Nonlin. 11, 37-90 (1994)

[16] V.V. Bolotin, *Nonconservative problems in the theory of elastic stability*, Fizmatgiz, Moscow, 1961. Pergamon, London, 1963

[17] V.V. Bolotin, N.I. Zhinzher, *Effect of damping on stability of elastic systems to nonconservative forces*, Int. J. Solids Struct. 5, 965-989 (1969)

[18] D. Bonheure, F. Gazzola, E. Moreira dos Santos, *Periodic solutions and torsional instability in a nonlinear nonlocal plate equation*, SIAM J. Math. Anal. 51, 3052-3091 (2019)

[19] T. Cazenave, A. Haraux, F.B. Weissler, *A class of nonlinear, completely integrable abstract wave equations*, J. Dynam. Differential Equations 5, 129-154 (1993)

[20] I. Chueshov, I. Lasiecka, *Von Kármán evolution equations. Well-posedness and long-time dynamics*, Springer Monographs in Mathematics, Springer, New York (2010)

[21] P.G. Ciarlet, P. Rabier, *Les équations de von Kármán*, Studies in Mathematics and its Applications 27, Springer, Berlin (1980)

[22] N.J. Delatte, *Beyond failure*, ASCE Press (2009)

[23] F.B. Farquharson, *Letter to the Editor*, ENR p.37, July 3 (1941)

[24] V. Ferreira, F. Gazzola, E. Moreira dos Santos, *Instability of modes in a partially hinged rectangular plate*, J. Differential Equations 261, 6302-6340 (2016)

[25] A. Ferrero, F. Gazzola, *A partially hinged rectangular plate as a model for suspension bridges*, Disc. Cont. Dynam. Syst. A 35, 5879-5908 (2015)

[26] M. Garrione, F. Gazzola, *Loss of energy concentration in nonlinear evolution beam equations*, J. Nonlinear Science 27, 1789-1827 (2017)

[27] M. Garrione, F. Gazzola, *Nonlinear equations for beams and degenerate plates with piers*, Springer Briefs (2019)

[28] M. Garrione, F. Gazzola, *Linear theory for beams with intermediate piers*, Comm. Contemp. Math. 22, 1950081, 41 pp. (2020)

[29] F. Gazzola, *Mathematical models for suspension bridges*, MS&A Vol. 15, Springer (2015)

[30] F. Gazzola, Y. Wang, *Modeling suspension bridges through the von Kármán quasilinear plate equations*, Progress in Nonlinear Differential Equations and Their Applications, In: Contributions to Nonlinear Differential Equations and Systems, a tribute to Djairo Guedes de Figueiredo on occasion of his 80th birthday, 269-297 (2015)

[31] D. Green, W.G. Unruh, *Tacoma Bridge failure - a physical model*, Amer. J. Physics 74, 706-716 (2006)

[32] Z-J. Han, E. Zuazua, *Decay rates for 1-d heat-wave planar networks*, Netw. Heterog. Media 11, 655-692 (2016)

[33] A. Haraux, *Nonlinear vibrations and the wave equation*, Springer Briefs in Mathematics, BCAM Springer, Cham (2018)

[34] A. Haraux, *A sharp stability criterion for single well Duffing and Duffing-like equations*, Nonlinear Anal. 190, 111600, 9 pp. (2020)

[35] B.V. Kapitonov, *Uniform stabilization and exact controllability for a class of coupled hyperbolic systems*, Mat. Apl. Comput. 15, 199-212 (1996)

[36] O.N. Kirillov, *Destabilization paradox*, Doklady Physics 49, 239-245 (2004). Translated from Doklady Akademii Nauk 395, 614-620 (2004)

[37] R. Krechetnikov, J.E. Marsden, *Dissipation-induced instabilities in finite dimensions*, Rev. Modern. Physics 79, 519-553 (2007)

[38] R. Krechetnikov, J.E. Marsden, *Dissipation-induced instability phenomena in infinite-dimensional systems*, Arch. Rational Mech. Anal. 194, 611-668 (2009)

[39] A.C. Lazer, P.J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. 32, 537-578 (1990)

[40] P.J. McKenna, *Torsional oscillations in suspension bridges revisited: fixing an old approximation*, Amer. Math. Monthly 106, 1-18 (1999)

[41] P.J. McKenna, *Oscillations in suspension bridges, vertical and torsional*, Disc. Cont. Dynam. System S 7, 785-791 (2014)

[42] New York Times, *A great bridge falls*, November 9, 1940

[43] S.A. Messaoudi, A. Soufyane, *Boundary stabilization of solutions of a nonlinear system of Timoshenko type*, Nonlinear Analysis 67, 2107-2121 (2007)

[44] J.E. Munoz Rivera, A. Soufyane, M.L. Santos, *General decay for full von Karman system with memory*, Nonlinear Analysis: Real World Applications 13, 2633-2647 (2012)

[45] R.H. Plaut, F.M. Davis, *Sudden lateral asymmetry and torsional oscillations of section models of suspension bridges*, J. Sound and Vibration 307, 894-905 (2007)

[46] J. Rauch, X. Zhang, E. Zuazua, *Polynomial decay of a hyperbolic-parabolic coupled system*, J. Math. Pures Appl. 84, 407-470 (2005)

[47] D. Russell, *A general framework for the study of indirect damping mechanisms in elastic systems*, J. Math. Anal. Appl. 173, 339-354 (1993)

[48] R.H. Scanlan, *The action of flexible bridges under wind, II: buffeting theory*, J. Sound and Vibration 60, 201-211 (1978)

[49] R.H. Scanlan, J.J. Tomko, *Airfoil and bridge deck flutter derivatives*, J. Eng. Mech. 97, 1717-1737 (1971)

[50] R. Scott, *In the wake of Tacoma. Suspension bridges and the quest for aerodynamic stability*, ASCE, Reston (2001)

[51] J.J. Sinou, L. Jezequel, *Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping*, European J. Mechanics - A/Solids, 26 (1), 106-122 (2007)

[52] A. Soufyane, *Uniform stability of coupled second order equations*, Electron. J. Diff. Eqns 25, 1-10 (2001)

[53] A.F. Vakakis, *Inducing passive nonlinear energy sinks in vibrating systems*, Transactions of the ASME 123, 324-332 (2001)

[54] T. von Kármán, *Festigkeitsprobleme im maschinenbau*, Encycl. der Mathematischen Wissenschaften, Leipzig, IV/4 C, 348-352 (1910)

[55] YouTube, *Tacoma Narrows Bridge collapse*, `http://www.youtube.com/watch?v=3mclp9QmCGs` (1940)

[56] X. Zhang, E. Zuazua, *Long time behavior of a coupled heat-wave system arising in fluid-structure interaction*, Arch. Rat. Mech. Anal. 184, 49-120 (2007)

[57] H. Ziegler, *Die stabilitätskriterien der elastomechanik*, Ing. Archiv. 20, 49-56 (1952)

[58] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM J. Control Optim. 28, 466-477 (1990)