GLOBAL EXISTENCE VERSUS BLOW-UP RESULTS FOR A FOURTH ORDER PARABOLIC PDE INVOLVING THE HESSIAN

CARLOS ESCUDERO, FILIPPO GAZZOLA, IRENEO PERAL

ABSTRACT. We consider a partial differential equation that arises in the coarse-grained description of epitaxial growth processes. This is a parabolic equation whose evolution is dictated by the competition among the determinant of the Hessian matrix of the solution and the biharmonic operator. This model might present a gradient flow structure depending on the boundary conditions. We first extend previous results on the existence of stationary solutions to this model for Dirichet boundary conditions. For the evolution problem we prove local existence of solutions for arbitrary data and global existence of solutions for small data. Depending on the boundary conditions and the concomitant presence of a variational structure in the equation as well as on the size of the data we prove blow-up of the solution in finite time and convergence to a stationary solution in the long time limit.

1. INTRODUCTION

Epitaxial growth is a technique by means of which the deposition of new material on existing layers of the same material takes place under high vacuum conditions. It is used in the semiconductor industry for the growth of crystalline structures that might be composed of pure chemical elements like silicon or germanium, or it could instead be formed by alloys like gallium arsenide or indium phosphide. In the case of molecular beam epitaxy the deposition is a very slow process and happens almost atom by atom.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded smooth domain which is the place where the deposition takes place. Although this kind of mathematical model can be studied in any spatial dimension N, we will concentrate here on the physical situation $N = 2$. The macroscopic evolution of the growth process can be modeled with a partial differential equation that is frequently proposed invoking phenomenological and symmetry arguments [4, 26]. The solution of such a differential equation is the function

$$u : \Omega \times \mathbb{R}_+ \to \mathbb{R},$$

describing the height of the growing interface at the spatial location $x \in \Omega$ at the temporal instant $t \in \mathbb{R}_+ := [0, \infty)$. A fundamental modeling assumption in this

Supported by MTM2010-18128, RYC-2011-09025, SEV-2011-0087, MIUR-PRIN-2008.

Date: October 8, 2013.

Key words and phrases. Epitaxial growth, higher order parabolic equations, global solutions, blow-up in finite time, potential well.

2010 *MSC: 35J30, 35K25, 35K35, 35K55, 35G31, 35Q70.*

field is considering that the physical interface can be described as the graph of u, and this is a valid hypothesis in an important number of cases [4].

One of the most widespread examples of this type of theory is the Kardar-Parisi-Zhang equation [20]

$$u_t = \nu \Delta u + \gamma |\nabla u|^2 + \eta(x,t),$$

which has been extensively studied in the physical literature and it is currently being investigated for its interesting mathematical properties [1, 2]. On the other hand, it has been argued that epitaxial growth processes should be described by a different equation coming from a conservation law and, in particular, the term $|\nabla u|^2$ should not be present in such a model [4]. An equation fulfilling these properties is the conservative counterpart of the Kardar-Parisi-Zhang equation [23, 34, 36]

$$u_t = -\mu \Delta^2 u + \kappa \Delta |\nabla u|^2 + \zeta(x,t). \tag{1}$$

This equation is conservative in the sense that the mean value $\int_\Omega u\,dx$ is constant if boundary conditions that isolate the system are used. It can also be considered as a higher order counterpart of the Kardar-Parisi-Zhang equation. In recent years, much attention has been devoted to other models of epitaxial growth, see [18, 21, 22, 25] and references therein.

Herein we will consider a different model obtained by means of the variational formulation developed in [26] and aimed at unifying previous approaches. We skip the detailed derivation of our model, that can be found in [8], and move to the resulting equation, that reads

$$u_t = 2\,K_1\,\det\left(D^2 u\right) - K_2\,\Delta^2 u + \xi(x,t).$$

This partial differential equation can be thought of as an analogue of equation (1); in fact, they are identical from a strict dimensional analysis viewpoint. Let us also note that this model has been shown to constitute a suitable description of epitaxial growth in the same sense as equation (1), and it even displays more intuitive geometric properties [7, 10]. The constants K_1 and K_2 will be rescaled in the following.

In this work we are interested in the following initial-boundary value problem:

$$\begin{cases} u_t + \Delta^2 u = \det(D^2 u) + \lambda f & x \in \Omega\,,\ t>0\,, \\ u(x,0) = u_0(x), & x \in \Omega\,, \\ \text{boundary conditions} & x \in \partial\Omega\,,\ t>0\,, \end{cases} \tag{2}$$

where f is some function possibly depending on both space and time coordinates and belonging to some Lebesgue space, $\lambda \in \mathbb{R}$. The initial condition $u_0(x)$ is also assumed to belong to some Sobolev space. We will consider the following sets of boundary conditions

$$u = u_\nu = 0, \qquad x \in \partial\Omega,$$

which we will refer to as Dirichlet boundary conditions, and

$$u = \Delta u = 0, \qquad x \in \partial\Omega,$$

which we will refer to as Navier boundary conditions. We note that the stationary solutions to this model were studied before [8, 9, 11].

For the evolution problem (2) we prove existence of a solution, both for arbitrary time intervals and small data, and for arbitrary data and small time intervals. Then using several tools from both critical point theory and potential well techniques, we prove the existence of finite time blow up solutions as well as the existence of global in time solutions, in suitable functional spaces. The use of these tools is by far nontrivial both because the nonlinearity occurs in the second order derivatives and because more regularity is necessary to overcome some delicate technical points.

This paper is organized as follows. In Section 2 we extend previous results in [11] concerning the stationary problem with Dirichlet boundary conditions and characterize the geometry of the functional that allows the variational treatment of this problem. In Section 3 we build the existence theory for the parabolic problem with both sets of boundary conditions and the presence of a source term. Section 4 is devoted to the analysis of the long time behavior and the blow-up in finite time of the solutions to the Dirichlet problem in the absence of a source term; this analysis is carried out taking advantage of the gradient flow structure of the equation in this case and of the so-called potential well techniques. Finally, in Section 5 we present some further results, including the proof of finite time blow-up of the solutions to the Navier problem for large enough initial conditions, and propose some open questions.

2. The stationary problem

2.1. **Existence of solutions with Dirichlet conditions.** In the sequel, we need several different norms. All the norms in $W^{s,p}$-spaces will be reported explicitly (that is, $\|\cdot\|_{W^{s,p}(\Omega)}$) except for the L^p-norm and the $W_0^{2,2}$-norm, respectively denoted by

$$(1 \le p < \infty) \quad \|u\|_p^p = \int_\Omega |u|^p \,, \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \,,$$

$$\|u\|^2 = \|\Delta u\|_2^2 = \int_\Omega |\Delta u|^2 \,,$$

We start by focusing on the following nonhomogeneous problem

$$\begin{cases} \Delta^2 u = \det(D^2 u) + f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ u_\nu = h & \text{on } \partial\Omega \end{cases} , \tag{3}$$

where $f \in L^1(\Omega)$, $g \in W^{3/2,2}(\partial\Omega)$, $h \in H^{1/2,2}(\partial\Omega)$. The following result holds.

Theorem 2.1. *There exists* $\gamma > 0$ *such that if*

$$\|f\|_1 + \|g\|_{W^{3/2,2}(\partial\Omega)} + \|h\|_{W^{1/2,2}(\partial\Omega)} < \gamma \tag{4}$$

then (3) *admits at least two weak solutions in* $W^{2,2}(\Omega)$*, a stable solution and a mountain pass solution.*

Proof. Consider the auxiliary linear problem

$$\begin{cases} \Delta^2 v = f & \text{in } \Omega \\ v = g & \text{on } \partial\Omega \\ v_\nu = h & \text{on } \partial\Omega \end{cases} . \tag{5}$$

In view of the embedding $L^1(\Omega) \subset W^{-2,2}(\Omega)$, [16, Theorem 2.16] tells us that (5) admits a unique weak solution $v \in W^{2,2}(\Omega)$ which satisfies

$$\|D^2 v\|_2 \le C\Big(\|f\|_1 + \|g\|_{W^{3/2,2}(\partial\Omega)} + \|h\|_{W^{1/2,2}(\partial\Omega)}\Big) \tag{6}$$

for some $C > 0$ independent of f, g, h. Subtracting (5) from (3) and putting $w = u - v$ we get

$$\begin{cases} \Delta^2 w = \det[D^2(w+v)] & \text{in } \Omega \\ w = w_\nu = 0 & \text{on } \partial\Omega \end{cases} .$$

This problem can be written as

$$\begin{cases} \Delta^2 w = \det(D^2 w) + \det(D^2 v) + v_{xx} w_{yy} + w_{xx} v_{yy} - 2 w_{xy} v_{xy} & \text{in } \Omega \\ w = w_\nu = 0 & \text{on } \partial\Omega \end{cases} . \tag{7}$$

By combining results from [5, 6, 27], Escudero-Peral [11] proved that for all $u \in W_0^{2,2}(\Omega)$ one has that $\det(D^2 u)$ belongs to the Hardy space and that

$$\det(D^2 u) = \Big(u_x u_{yy}\Big)_x - \Big(u_x u_{xy}\Big)_y = \Big(u_x u_y\Big)_{xy} - \frac{1}{2}\Big(u_y^2\Big)_{xx} - \frac{1}{2}\Big(u_x^2\Big)_{yy}$$

in $\mathcal{D}'(\Omega)$. Moreover,

$$\int_\Omega u \det(D^2 u) = 3 \int_\Omega u_x u_y u_{xy} \quad \forall u \in W_0^{2,2}(\Omega). \tag{8}$$

These facts show that (7) admits a variational formulation. The corresponding functional reads

$$K(w) = \int_\Omega \left[\frac{|\Delta w|^2}{2} - w_x w_y w_{xy} - \det(D^2 v) w + \frac{w_y^2 v_{xx}}{2} + \frac{w_x^2 v_{yy}}{2} - w_x w_y v_{xy} \right].$$

Note that, by the embedding $W_0^{2,2}(\Omega) \subset W_0^{1,4}(\Omega)$, we have

$$K(w) \ge -\int_\Omega \big[w_x w_y w_{xy} + \det(D^2 v) w\big] + \frac{1}{2}\|\Delta w\|_2^2 - C\|D^2 v\|_2 \|\Delta w\|_2^2,$$

so a mountain pass geometry [3] is ensured for small enough $\|D^2 v\|_2$. In view of (6), the mountain pass geometry is ensured if γ in (4) is sufficiently small. This geometry yields the existence of a locally minimum solution and of a mountain pass solution. □

Theorem 2.1 generalizes the following statement proved in [11]:

Corollary 2.2. *The Dirichlet problem*

$$\begin{cases} \Delta^2 u = \det(D^2 u) & \textit{in } \Omega \\ u = u_\nu = 0 & \textit{on } \partial\Omega \end{cases} \tag{9}$$

admits a nontrivial weak solution $u \in W_0^{2,2}(\Omega)$.

Concerning the regularity of solutions, we have the following statement.

Theorem 2.3. *Assume that, for some integer $k \geq 0$ we have: $\partial\Omega \in C^{k+4}$, $f \in W^{k,2}(\Omega)$, $g \in W^{k+7/2,2}(\partial\Omega)$, $h \in W^{k+5/2,2}(\partial\Omega)$. Then any solution to* (3) *satisfies*

$$u \in W^{k+4,2}(\Omega) \, .$$

In particular, any solution to (9) *is as smooth as the boundary permits.*

Proof. By duality, from the embedding $W_0^{s,2}(\Omega) \subset L^\infty(\Omega)$ we infer that $L^1(\Omega) \subset [L^\infty(\Omega)]' \subset W^{-s,2}(\Omega)$ for all $s > 1$. Therefore, for any solution $u \in W^{2,2}(\Omega)$ to (3) we have $\det(D^2u) \in W^{-s,2}(\Omega)$ for all $s > 1$. Therefore, even if $k = 0$, we have $\Delta^2 u \in W^{-s,2}(\Omega)$ and, in turn, $u \in W^{r,2}(\Omega)$ for any $r < 3$. A bootstrap argument and elliptic regularity then allow to conclude. □

Remark 2.4. If we stop the previous proof at the first step, we see that, in a C^3 domain, any solution to

$$\begin{cases} \Delta^2 u = \det(D^2u) + f & \text{in } \Omega \\ u = u_\nu = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$ belongs to $W^{r,2}(\Omega)$ for any $r < 3$, which slightly improves the result in [11]. Note also that these arguments take strong advantage of being in planar domains.

2.2. **The Nehari manifold and the mountain pass level.** The energy functional for the stationary problem (9) is

$$J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 - \int_\Omega v_x v_y v_{xy} \qquad \forall v \in W_0^{2,2}(\Omega). \tag{10}$$

It is shown in [11] that J has a mountain pass geometry and that the corresponding mountain pass level is given by

$$d \; = \; \inf_{\gamma\in\Gamma} \; \max_{0\leq s\leq 1} \; J(\gamma(s)) \tag{11}$$

where $\Gamma := \{\gamma \in C([0,1], W_0^{2,2}(\Omega));\ \gamma(0) = 0,\ J(\gamma(1)) < 0\}$. We aim to characterize differently d and to relate it with the so-called Nehari manifold defined by

$$\mathcal{N} := \Big\{ v \in W_0^{2,2}(\Omega) \setminus \{0\};\ \langle J'(v), v\rangle = \|v\|^2 - 3\int_\Omega v_x v_y v_{xy} = 0 \Big\}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{2,-2}(\Omega)$ and $W_0^{2,2}(\Omega)$. To this end, we introduce the set

$$B := \{v \in W_0^{2,2}(\Omega);\ \int_\Omega v_x v_y v_{xy} = 1\} \, . \tag{12}$$

It is clear that $v \in \mathcal{N}$ if and only if $\alpha v \in B$ for some $\alpha > 0$. In particular, not on all the straight directions starting from 0 in the phase space $W_0^{2,2}(\Omega)$ there exists an intersection with $\mathcal{N}$. Hence, $\mathcal{N}$ is an unbounded manifold (of codimension 1) which separates the two regions

$$\mathcal{N}_+ = \left\{ v \in W_0^{2,2}(\Omega);\ \|v\|^2 > 3\int_\Omega v_x v_y v_{xy} \right\}$$

and

$$\mathcal{N}_- = \left\{ v \in W_0^{2,2}(\Omega);\ \|v\|^2 < 3\int_\Omega v_x v_y v_{xy} \right\} .$$

The next result states some properties of $\mathcal{N}_\pm$.

Theorem 2.5. *Let $v \in W_0^{2,2}(\Omega)$, then the following implications hold:*
(i) $0 < \|v\|^2 < 6d \implies v \in \mathcal{N}_+$;
(ii) $v \in \mathcal{N}_+,\ J(v) < d \implies 0 < \|v\|^2 < 6d$;
(iii) $v \in \mathcal{N}_- \implies \|v\|^2 > 6d$.

Proof. It is well-known [3] that the mountain pass level d may also be defined by

$$d = \min_{v\in\mathcal{N}} J(v) . \tag{13}$$

Using (13) and the definition of $\mathcal{N}$ we obtain

$$d = \min_{v\in\mathcal{N}} J(v) = \min_{v\in\mathcal{N}} \left(\frac{\|v\|^2}{2} - \int_\Omega v_x v_y v_{xy} \right) = \min_{v\in\mathcal{N}} \frac{\|v\|^2}{6}$$

which proves (i) since $\mathcal{N}$ separates $\mathcal{N}_+$ and $\mathcal{N}_-$.

If $v \in \mathcal{N}_+$, then $-\int_\Omega v_x v_y v_{xy} > -\|v\|^2/3$. If $J(v) < d$, then $\|v\|^2 - 2\int_\Omega v_x v_y v_{xy} < 2d$. By combining these two inequalities we obtain (ii).

Finally, recalling the definitions of $\mathcal{N}_\pm$, (iii) follows directly from (i). □

A further functional needed in the sequel is given by

$$I(v) = \int_\Omega v_x v_y v_{xy} . \tag{14}$$

We provide a different characterization of the mountain pass level.

Theorem 2.6. *The mountain pass level d for J is also determined by*

$$d = \min_{v\in B} \frac{\|v\|^6}{54} . \tag{15}$$

Moreover, d can be lower bounded in terms of the best constant for the (compact) embedding $W_0^{2,2}(\Omega) \subset W_0^{1,4}(\Omega)$, namely

$$d \geq \frac{8}{27} \min_{W_0^{2,2}(\Omega)} \frac{(\int_\Omega |\Delta v|^2)^2}{\int_\Omega |\nabla v|^4} .$$

Proof. For all $v \in W_0^{2,2}(\Omega)$ consider the map $f_v : [0,+\infty) \to \mathbb{R}$ defined by

$$f_v(s) = J(sv) = \frac{s^2}{2}\int_\Omega |\Delta v|^2 - s^3 \int_\Omega v_x v_y v_{xy} .$$

If $I(v) \leq 0$, the map $s \mapsto f_v(s)$ is strictly increasing and strictly convex, attaining its global minimum at $s = 0$; in this case, f_v has no critical points apart from $s = 0$.

So, the mountain pass level is achieved for some function v satisfying $I(v) > 0$. For any $v \in B$, see (12), we have

$$f_v(s) = \frac{\|v\|^2}{2}s^2 - s^3 \ .$$

It is straightforward to verify that the map $s \mapsto f_v(s)$ is initially increasing and then strictly decreasing. It attains the global maximum for $s = \frac{\|v\|^2}{3}$ and

$$\max_{s \geq 0} f_v(s) = \frac{\|v\|^6}{54} \ .$$

Hence,

$$\max_{s \geq 0} J(sv) = \frac{\|v\|^6}{54} \qquad \forall v \in B \ .$$

By the minimax characterization of the mountain pass level we see that (15) holds.

Next, note that integrating by parts we obtain

$$I(v) = \frac{1}{2}\int_\Omega v_x (v_y^2)_x = -\frac{1}{2}\int_\Omega v_{xx}\, v_y^2 = -\frac{1}{2}\int_\Omega \Delta v\, v_y^2 + \frac{1}{6}\int_\Omega (v_y^3)_y \ ,$$

$$I(v) = \frac{1}{2}\int_\Omega v_y (v_x^2)_y = -\frac{1}{2}\int_\Omega v_{yy}\, v_x^2 = -\frac{1}{2}\int_\Omega \Delta v\, v_x^2 + \frac{1}{6}\int_\Omega (v_x^3)_x \ ,$$

$$\forall v \in W_0^{2,2}(\Omega) \ .$$

Hence, by adding and by the divergence Theorem,

$$I(v) = -\frac{1}{4}\int_\Omega \Delta v\, |\nabla v|^2 + \frac{1}{12}\int_\Omega [(v_x^3)_x + (v_y^3)_y] = -\frac{1}{4}\int_\Omega \Delta v\, |\nabla v|^2$$

$$\forall v \in W_0^{2,2}(\Omega) \ . \tag{16}$$

Therefore, by Hölder inequality,

$$I(v) \leq \frac{1}{4}\left(\int_\Omega |\Delta v|^2\right)^{1/2}\left(\int_\Omega |\nabla v|^4\right)^{1/2} \qquad \forall v \in W_0^{2,2}(\Omega) \tag{17}$$

and, according to (15), we infer

$$d = \frac{1}{54}\min_X \frac{(\int_\Omega |\Delta v|^2)^3}{I(v)^2} \geq \frac{8}{27}\min_{W_0^{2,2}(\Omega)} \frac{(\int_\Omega |\Delta v|^2)^2}{\int_\Omega |\nabla v|^4}$$

where $X := \{v \in W_0^{2,2}(\Omega);\ I(v) > 0\}$. □

In Figure 2.1 we sketch a geometric representation of the Nehari manifold $\mathcal{N}$ which summarizes the results obtained in the present section.

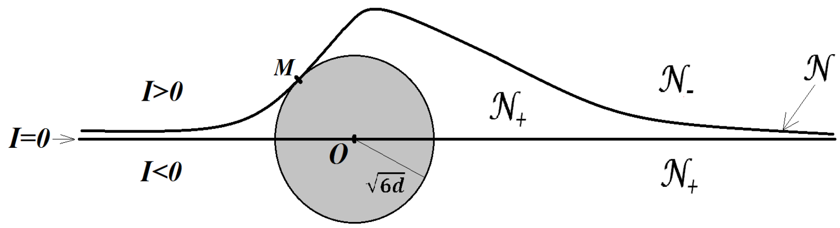

FIGURE 2.1. The phase space $W_0^{2,2}(\Omega)$ with: $\mathcal{N}$ = Nehari manifold, M = mountain pass point, and I given by (14).

3. THE PARABOLIC PROBLEM WITH SOURCE

This section is devoted to the study of the evolution problem

$$u_t + \Delta^2 u = \det(D^2 u) + \lambda f \qquad \text{in } \Omega \times (0,T) \tag{18}$$

for some $T > 0$. We consider both the sets of boundary conditions $u|_{\partial\Omega} = u_\nu|_{\partial\Omega} = 0$ (Dirichlet) and $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$ (Navier). Here and in the sequel we will be always considering weak solutions.

We start by proving a result concerning an associated linear problem.

Theorem 3.1. *Let $0 < T \le \infty$ and let $f \in L^2(0,T;L^2(\Omega))$. The Dirichlet problem for the linear fourth order parabolic equation*

$$u_t + \Delta^2 u = f \qquad \text{in } \Omega \times (0,T), \tag{19}$$

with initial datum $u_0 \in W_0^{2,2}(\Omega)$ admits a unique weak solution in the space

$$C(0,T;W_0^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)).$$

The corresponding Navier problem with initial datum $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ admits a unique weak solution in the space

$$C(0,T;W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)).$$

Furthermore, both cases admit the estimate

$$\max_{0\le t\le T} \|\Delta u\|_2^2 + \int_0^T \|\Delta^2 u\|_2^2 + \int_0^T \|u_t\|_2^2 \le C \left(\|\Delta u_0\|_2^2 + \int_0^T \|f\|_2^2 \right) .$$

Proof. STEP 1. EXISTENCE VIA GALERKIN METHOD. We will focus herein on Dirichlet boundary conditions; the proof for the Navier problem follows with obvious modifications. Let $u_0 \in W_0^{2,2}(\Omega)$ and consider the following linear problem

$$\begin{cases} u_t + \Delta^2 u = f & \text{in } \Omega \times (0,T) \\ u = u_\nu = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega\,. \end{cases} \tag{20}$$

Let $\{w_k\}_{k\ge1} \subset W_0^{2,2}(\Omega)$ be an orthogonal complete system of eigenfunctions of Δ^2 under Dirichlet boundary conditions normalized by $\|w_k\|_2 = 1$. Denote by $\{\lambda_k\}$ the unbounded sequence of corresponding eigenvalues and by

$$W_k := \text{span}\{w_1, \dots, w_k\} \qquad \forall k \ge 1.$$

Denote by $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ the scalar products in $L^2(\Omega)$ and $W^{2,2}_0(\Omega)$. For any $k\geq 1$ let

$$u_0^k := \sum_{i=1}^k (u_0, w_i)_2 w_i = \sum_{i=1}^k \lambda_i^{-1}(u_0, w_i)\, w_i$$

so that $u_0^k \to u_0$ in $W^{2,2}_0(\Omega)$ as $k\to+\infty$. For any $k\geq 1$ we seek a solution $u_k\in W^{1,2}(0,T;W_k)$ of the variational problem

$$\left\{\begin{array}{l}(u'(t),v)_2 + (u(t),v) = (f(t),v)_2\\ \qquad \text{for any } v\in W_k \quad \text{for a.e. } t\in(0,T)\\ u(0)=u_0^k\,.\end{array}\right. \tag{21}$$

We seek solutions in the form

$$u_k(t) = \sum_{i=1}^k g_i^k(t) w_i$$

so that for any $1\leq i\leq k$ the function g_i^k solves the Cauchy problem

$$\left\{\begin{array}{l}(g_i^k(t))' + \lambda_i g_i^k(t) = (f(t),w_i)_{L^2(\Omega)}\\ g_i^k(0) = (u_0^k, w_i)_{L^2(\Omega)}\,.\end{array}\right. \tag{22}$$

The linear ordinary differential equation (22) admits a unique solution g_i^k such that $g_i^k\in W^{1,2}(0,T)$, and hence also (21) admits $u_k\in W^{1,2}(0,T;W_k)$ as a unique solution.

Note that

$$\Delta^2 u_k(t) = \sum_{i=1}^k g_i^k(t)\lambda_i w_i \in W_k \qquad \text{for a.e. } t\in(0,T)$$

so that by testing equation (21) with $v=\Delta^2 u_k(t)$ we obtain that for a.e. $t\in(0,T)$:

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|^2 + \frac{1}{2}\|u_k(t)\|^2_{W^{4,2}(\Omega)} = (f(t),\Delta^2 u_k(t))_2.$$

After integration over $(0,t)$ we obtain

$$\|u_k(t)\|^2 - \|u_0^k\|^2 + \|u_k\|^2_{L^2(0,t;W^{4,2}(\Omega))} \leq \int_0^T \left(C\|f(s)\|_2^2 + \frac{1}{2}\|u_k(s)\|^2_{W^{4,2}(\Omega)}\right) ds$$

and therefore

$$\|u_k\|^2_{L^\infty(0,T;W^{2,2}_0(\Omega))} + \frac{1}{2}\|u_k\|^2_{L^2(0,T;W^{4,2}(\Omega))} \leq \|u_0^k\|^2 + C\|f\|^2_{L^2(0,T;L^2(\Omega))}.$$

Since the sequence $\{u_0^k\}$ is bounded in $W^{2,2}_0(\Omega)$, we infer that

$$\{u_k\} \quad \text{is bounded in} \quad L^\infty(0,T;W^{2,2}_0(\Omega))\cap L^2(0,T;W^{4,2}(\Omega)).$$

Whence, we may extract a subsequence, still denoted by $\{u_k\}$ such that

$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T;W^{2,2}_0(\Omega)) \qquad \text{and} \qquad u_k \rightharpoonup u \text{ in } L^2(0,T;W^{4,2}(\Omega))\,.$$

Moreover, since $u_k' = -\Delta^2 u_k + f$ in the weak sense, we also have that $u_k' \in L^2(0,T;L^2(\Omega))$ and that

$$u_k' \rightharpoonup u' \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence, by letting $k \to \infty$ in (21), we see that

$$u \in L^\infty(0,T;W_0^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$$

solves the problem

$$\left\{ \begin{array}{l} (u'(t),v)_2 + (u(t),v) = (f(t),v)_2 \\ \qquad \text{for any } v \in W_0^{2,2}(\Omega) \quad \text{for a.e. } t \in (0,T) \\ u(0) = u_0\,. \end{array} \right. \tag{23}$$

By interpolation between $L^2(0,T;W^{4,2}(\Omega))$ and $W^{1,2}(0,T;L^2(\Omega))$ we obtain $u \in C(0,T;W_0^{2,2}(\Omega))$.

STEP 2. ESTIMATES. The existence result justifies the following calculations performed in order to obtain the desired estimate. We multiply equation (19) by $\Delta^2 u$ and integrate by parts over Ω the result to find

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 + \|\Delta^2 u\|_2^2 = (\Delta^2 u, f)_2 \le \frac{\epsilon}{2}\|\Delta^2 u\|_2^2 + \frac{1}{2\epsilon}\|f\|_2^2 \quad \text{for a.e. } t \in (0,T)$$

for any $\epsilon > 0$. Upon integration in time we obtain

$$\max_{0\le t\le T}\|\Delta u\|_2^2 + \int_0^T \|\Delta^2 u\|_2^2 \le C\left(\|\Delta u_0\|_2^2 + \int_0^T \|f\|_2^2\right).$$

To conclude multiply equation (19) by an arbitrary function $v \in L^2(\Omega)$ to get

$$(v,u_t)_2 + (v,\Delta^2 u)_2 = (v,f)_2 \quad \text{for a.e. } t \in (0,T).$$

This equality implies the inequality

$$(v,u_t)_2 \le \|f\|_2\|v\|_2 + \|\Delta^2 v\|_2\|v\|_2.$$

Now taking the supremum over all $v \in L^2(\Omega)$ such that $\|v\|_2 = 1$ and the fact

$$\sup_v (v,u_t)_2 = \|u_t\|_2,$$

we find

$$\int_0^T \|u_t\|_2^2 \le C\left(\int_0^T \|\Delta^2 u\|_2^2 + \int_0^T \|f\|_2^2\right),$$

and the desired inequality follows immediately.

Finally, uniqueness follows by a standard contradiction argument. □

Now we state the main result of this section.

Theorem 3.2. *The problem*

$$\left\{ \begin{array}{ll} u_t + \Delta^2 u = \det(D^2 u) + \lambda f & \textit{in } \Omega \times (0,T) \\ u(x,0) = u_0(x) & \textit{in } \Omega \\ u(x,t) = u_\nu(x,t) = 0 & \textit{on } \partial\Omega \times (0,T) \end{array} \right. \tag{24}$$

admits a unique solution in

$$\mathcal{X}_T := C(0,T;W_0^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)),$$

provided one of the following set of conditions holds

(i) $u_0 \in W_0^{2,2}(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, $\lambda \in \mathbb{R}$, and $T > 0$ is sufficiently small;

(ii) $T \in (0,\infty]$, $f \in L^2(0,T;L^2(\Omega))$, and $\|u_0\|$ and $|\lambda|$ are sufficiently small.

Moreover, if $[0,T^*)$ denotes the maximal interval of continuation of u and if $T^* < \infty$ then $\|u(t)\| \to \infty$ as $t \to T^*$.

An identical result holds for the Navier problem but this time the solution belongs to the space

$$\mathcal{Y}_T := C(0,T;W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)),$$

assuming that the initial condition $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Proof. For all $u \in W^{4,2}(\Omega)$ we have

$$\begin{aligned}\|\det(D^2u)\|_2^2 &= \int_\Omega |\det(D^2u)|^2 \le C\int_\Omega |D^2u|^4 \le C\|D^2u\|_\infty^2 \int_\Omega |D^2u|^2 \\ &\le C\|\Delta u\|_\infty^2\, \|\Delta u\|_2^2 \le C\|\Delta^2 u\|_2^2\, \|\Delta u\|_2^2.\end{aligned}$$

Hence, if $u \in C(0,T;W^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega))$, we may directly estimate

$$\begin{aligned}\|\det(D^2u)\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T \|\det(D^2u)\|_2^2 \le C\int_0^T \|\Delta^2u\|_2^2\, \|\Delta u\|_2^2 \\ &\le C\max_{0\le t\le T}\|\Delta u\|_2^2 \int_0^T \|\Delta^2 u\|_2^2 < \infty\end{aligned}$$

which proves that

$$u \in C(0,T;W^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \Longrightarrow \det(D^2u) \in L^2(0,T;L^2(\Omega)).$$

In what follows we focus on the Dirichlet case since the proof for the Navier one follows similarly. We introduce the initial-Dirichlet linear problems

$$\text{(25)}\qquad \begin{cases} (u_1)_t + \Delta^2 u_1 = \det(D^2v_1) + \lambda f\,, & u_1(x,0) = u_0(x)\,, \\ (u_2)_t + \Delta^2 u_2 = \det(D^2v_2) + \lambda f\,, & u_2(x,0) = u_0(x)\,, \end{cases}$$

where $v_1, v_2 \in \mathcal{X}_T$. The just proved inclusion and Theorem 3.1 show that $u_1, u_2 \in \mathcal{X}_T$. Subtracting the equations in (25) we get

$$(u_1-u_2)_t + \Delta^2(u_1-u_2) = \det(D^2v_1) - \det(D^2v_2)\,, \quad (u_1-u_2)(x,0) = 0\,,$$

and upon multiplying by $\Delta^2(u_1-u_2)$ and integrating we find

$$(\Delta^2(u_1-u_2),(u_1-u_2)_t)_2 + (\Delta^2(u_1-u_2),\Delta^2(u_1-u_2))_2 = (\Delta^2(u_1-u_2),\det(D^2v_1)-\det(D^2v_2))_2.$$

This leads to the inequalities

$$\frac{1}{2}\frac{d}{dt}\|\Delta(u_1-u_2)\|_2^2 + \|\Delta^2(u_1-u_2)\|_2^2 \le \frac{1}{2}\|\Delta^2(u_1-u_2)\|_2^2 + \frac{1}{2}\|\det(D^2v_1)-\det(D^2v_2)\|_2^2$$

and, in turn,

$$\frac{d}{dt}\|\Delta(u_1-u_2)\|_2^2+\|\Delta^2(u_1-u_2)\|_2^2\le\|\det(D^2v_1)-\det(D^2v_2)\|_2^2. \tag{26}$$

We split the remaining part of the proof into three steps.

STEP 1. EXISTENCE FOR ARBITRARY TEMPORAL LAPSES.

We start focussing on the case $T<\infty$ and estimating the term containing the determinants

$$\|\det(D^2v_1)-\det(D^2v_2)\|_2^2\le C\int_\Omega|D^2(v_1-v_2)|^2(|D^2v_1|+|D^2v_2|)^2\le \tag{27}$$

$$C(\|\Delta v_1\|_\infty^2+\|\Delta v_2\|_\infty^2)\|\Delta(v_1-v_2)\|_2^2\le C(\|\Delta^2v_1\|_2^2+\|\Delta^2v_2\|_2^2)\|\Delta(v_1-v_2)\|_2^2,$$

to infer from (26)

$$\frac{d}{dt}\|\Delta(u_1-u_2)\|_2^2+\|\Delta^2(u_1-u_2)\|_2^2\le C(\|\Delta^2v_1\|_2^2+\|\Delta^2v_2\|_2^2)\|\Delta(v_1-v_2)\|_2^2.$$

Integrating with respect to time we obtain

$$\max_{0\le t\le T}\|\Delta(u_1-u_2)\|_2^2+\int_0^T\|\Delta^2(u_1-u_2)\|_2^2\le \tag{28}$$

$$C\max_{0\le t\le T}\|\Delta(v_1-v_2)\|_2^2\int_0^T(\|\Delta^2v_1\|_2^2+\|\Delta^2v_2\|_2^2).$$

Now consider a function $w\in L^2(\Omega)$ and the scalar product

$$(w,(u_1-u_2)_t)_2+(w,\Delta^2(u_1-u_2))_2=(w,\det(D^2v_1)-\det(D^2v_2))_2.$$

We have the estimate

$$(w,(u_1-u_2)_t)_2\le$$
$$\|w\|_2\|\Delta^2(u_1-u_2)\|_2+\|w\|_2\|\det(D^2v_1)-\det(D^2v_2)\|_2,$$

and taking the supremum of all $w\in L^2(\Omega)$ such that $\|w\|_2=1$ we get

$$\sup_w(w,(u_1-u_2)_t)_2\le\|\Delta^2(u_1-u_2)\|_2+\|\det(D^2v_1)-\det(D^2v_2)\|_2.$$

Therefore, from (27) we infer that

$$\|(u_1-u_2)_t\|_2^2\le$$
$$C\left[\|\Delta^2(u_1-u_2)\|_2^2+(\|\Delta^2v_1\|_2^2+\|\Delta^2v_2\|_2^2)\|\Delta(v_1-v_2)\|_2^2\right],$$

and consequently, by using (28),

$$\max_{0\le t\le T}\|\Delta(u_1-u_2)\|_2^2+\int_0^T\|\Delta^2(u_1-u_2)\|_2^2+\int_0^T\|(u_1-u_2)_t\|_2^2\le \tag{29}$$

$$C\max_{0\le t\le T}\|\Delta(v_1-v_2)\|_2^2\int_0^T(\|\Delta^2v_1\|_2^2+\|\Delta^2v_2\|_2^2).$$

On the space $\mathcal{X}_T$ we define the norm

$$\|u\|_{\mathcal{X}_T}^2:=\max_{0\le t\le T}\|\Delta u\|_2^2+\int_0^T\|\Delta^2u\|_2^2+\int_0^T\|u_t\|_2^2,$$

so that (29) reads

$$\|u_1 - u_2\|_{\mathcal{X}_T} \le C \left[\int_0^T (\|\Delta^2 v_1\|_2^2 + \|\Delta^2 v_2\|_2^2) \right]^{1/2} \|v_1 - v_2\|_{\mathcal{X}_T}. \tag{30}$$

Now consider the unique solution u_ℓ (see Theorem 3.1) to the linear problem

$$(u_\ell)_t + \Delta^2 u_\ell = \lambda f,$$

with the same boundary and initial conditions as (25). Then define the ball

$$B_\rho = \{u \in \mathcal{X}_T : \|u - u_\ell\|_{\mathcal{X}_T} \le \rho\}. \tag{31}$$

Using estimate (30) we find

$$\|u_i - u_\ell\|_{\mathcal{X}_T} \le C \left(\int_0^T \|\Delta^2 v_i\|_2^2 \right)^{1/2} \|v_i\|_{\mathcal{X}_T} \le C \|v_i\|_{\mathcal{X}_T}^2, \tag{32}$$

for $i = 1, 2$. We use the triangle inequality

$$\|v_i\|_{\mathcal{X}_T} \le \|v_i - u_\ell\|_{\mathcal{X}_T} + \|u_\ell\|_{\mathcal{X}_T} \tag{33}$$

together with (see Theorem 3.1)

$$\|u_\ell\|_{\mathcal{X}_T}^2 \le C \left(\|\Delta u_0\|_2^2 + \lambda^2 \int_0^T \|f\|_2^2 \right) =: C\,\Gamma(\rho, u_0, \lambda, f). \tag{34}$$

to infer from (32)-(33)-(34) that

$$\|u_i - u_\ell\|_{\mathcal{X}_T} \le C \left(\rho^2 + \|\Delta u_0\|_2^2 + \lambda^2 \int_0^T \|f\|_2^2 \right),$$

and thus

$$\|u_i - u_\ell\|_{\mathcal{X}_T} \le \rho,$$

for small enough ρ, $|\lambda|$ and $\|\Delta u_0\|_2$.

By using (32)-(33)-(34) and reasoning as before we can transform (30) into

$$\|u_1 - u_2\|_{\mathcal{X}_T} \le C\,\Gamma(\rho, u_0, \lambda, f)^{1/2} \|v_1 - v_2\|_{\mathcal{X}_T}.$$

Again, for ρ, $|\lambda|$ and $\|\Delta u_0\|_2$ small enough we have

$$\|u_1 - u_2\|_{\mathcal{X}_T} \le \frac{1}{2} \|v_1 - v_2\|_{\mathcal{X}_T}.$$

The existence of a unique solution follows from the application of Banach fixed point theorem to the map

$$\begin{array}{rcl} \mathcal{A} : B_\rho & \to & B_\rho \\ v_i & \mapsto & u_i, \end{array}$$

for $i = 1, 2$. The case $T = \infty$ follows similarly since $\Gamma(\rho, u_0, \lambda, f)$ does not depend on how large is T.

STEP 2. LOCAL EXISTENCE IN TIME.

By the Gagliardo-Nirenberg inequality [13, 29],

$$\|\Delta v_i\|_\infty \le C \|\Delta v_i\|_2^{1/4} \|\nabla \Delta v_i\|_3^{3/4}, \qquad (i = 1, 2),$$

we may go back to (26) and we improve (27) with

$$\|\det(D^2v_1)-\det(D^2v_2)\|_2^2\le C(\|\Delta v_1\|_\infty^2+\|\Delta v_2\|_\infty^2)\|v_1-v_2\|^2\le$$
$$C(\|\Delta v_1\|_2^{1/2}\|\nabla\Delta v_1\|_3^{3/2}+\|\Delta v_2\|_2^{1/2}\|\nabla\Delta v_2\|_3^{3/2})\|v_1-v_2\|^2.$$

This, together with a Sobolev embedding, leads to

$$\frac{d}{dt}\|u_1-u_2\|^2+\|\Delta^2(u_1-u_2)\|_2^2\le$$
$$C(\|\Delta v_1\|_2^{1/2}\|\Delta^2 v_1\|_2^{3/2}+\|\Delta v_2\|_2^{1/2}\|\Delta^2 v_2\|_2^{3/2})\|v_1-v_2\|^2.$$

An integration with respect to time then yields

$$\max_{0\le t\le T}\|u_1-u_2\|^2+\int_0^T\|\Delta^2(u_1-u_2)\|_2^2\le C\max_{0\le t\le T}\|v_1-v_2\|^2\times$$
$$C\left(\max_{0\le t\le T}\|v_1\|^{1/2}\int_0^T\|\Delta^2v_1\|_2^{3/2}+\max_{0\le t\le T}\|v_2\|^{1/2}\int_0^T\|\Delta^2v_2\|_2^{3/2}\right).$$

We proceed making use of Hölder inequality to find

$$\max_{0\le t\le T}\|u_1-u_2\|^2+\int_0^T\|\Delta^2(u_1-u_2)\|_2^2\le C\,T^{1/4}\max_{0\le t\le T}\|v_1-v_2\|^2\times$$
$$\left[\max_{0\le t\le T}\|v_1\|^{1/2}\left(\int_0^T\|\Delta^2v_1\|_2^2\right)^{3/4}+\max_{0\le t\le T}\|v_2\|^{1/2}\left(\int_0^T\|\Delta^2v_2\|_2^2\right)^{3/4}\right].$$

Combining the estimates above with the arguments in Step 1 yields

$$\|u_1-u_2\|_{\mathcal{X}_T}\le C\,T^{1/4}\|v_1-v_2\|_{\mathcal{X}_T}\times$$
$$\left[\max_{0\le t\le T}\|v_1\|^{1/2}\left(\int_0^T\|\Delta^2v_1\|_2^2\right)^{3/4}+\max_{0\le t\le T}\|v_2\|^{1/2}\left(\int_0^T\|\Delta^2v_2\|_2^2\right)^{3/4}\right]^{1/2}.$$

Consider again the ball B_ρ defined in (31). In this case we have

$$\|u_i-u_\ell\|_{\mathcal{X}_T}\le C\,T^{1/4}\max_{0\le t\le T}\|v_i\|^{1/4}\left(\int_0^T\|\Delta^2v_i\|_2^2\right)^{3/8}\|v_i\|_{\mathcal{X}_T}$$
$$\le C\,T^{1/4}\|v_i\|_{\mathcal{X}_T}^2,$$

for $i=1,2$. Arguing as in Step 1 of the present proof we get

$$\|u_i-u_\ell\|_{\mathcal{X}_T}\le C\,T^{1/4}\Gamma(\rho,u_0,\lambda,f),$$

and thus

$$\|u_i-u_\ell\|_{\mathcal{X}_T}\le\rho,$$

for small enough T. Additionally we have

$$\|u_1-u_2\|_{\mathcal{X}_T}\le C\,T^{1/4}\Gamma(\rho,u_0,\lambda,f)^{1/2}\|v_1-v_2\|_{\mathcal{X}_T}.$$

Again, for T small enough we find

$$\|u_1-u_2\|_{\mathcal{X}_T}\le\frac{1}{2}\|v_1-v_2\|_{\mathcal{X}_T}.$$

The existence of a unique solution to (24) follows from the application of Banach fixed point theorem to the map

$$\begin{aligned} \mathcal{A} : B_\sigma &\to B_\sigma \\ v_i &\mapsto u_i \qquad (i = 1, 2). \end{aligned}$$

We have so found $\overline{T} = \overline{T}(\lambda, \|u_0\|)$ such that (24) admits a unique solution over $[0, T]$ for all $T < \overline{T}$.

STEP 3. BLOW-UP.

We argue by contradiction. Assume that $[0, T^*)$, with $T^* < \infty$, is the maximal interval of continuation of the solution, and that $\liminf_{t\to T^*} \|u(t)\| = \gamma < \infty$. Then there exists a sequence $\{t_n\}$ such that $t_n \to T^*$ and $\|u(t_n)\| < 2\gamma$ for n large enough. Take n sufficiently large so that $t_n + \overline{T}(\lambda, 2\gamma) > T^*$, where $\overline{T}$ is defined at the end of Step 2. Consider $u(t_n)$ as initial condition to (24). Then Step 2 tells us that the solution may be continued beyond T^*, contradiction. □

Corollary 3.3. *Let u be a solution as described in Theorem 3.2 during the time interval $(0, T]$. Then there exists a real number $\epsilon > 0$ such that the solution can be prolonged to the interval $(0, T + \epsilon]$.*

Proof. This result is a consequence of Step 3 in the proof of Theorem 3.2. □

It is possible to prove higher regularity of the solution if we neglect the source term.

Corollary 3.4. *Let u be a solution as described in Theorem 3.2 to equation* (18) *with $\lambda = 0$. Then $u^2 \in C^1(0, T; L^1(\Omega))$.*

Proof. The regularity proven in Theorem 3.2 for the solution u to (18) implies that $\det(D^2u) \in C(0, T; L^1(\Omega))$ and $\Delta^2 u \in C(0, T; W^{-2,2}(\Omega))$ so that

$$u_t = -\Delta^2 u + \det(D^2 u) \in C(0, T; W^{-2,2}(\Omega))$$

and, in turn, $u \in C^1(0, T; W^{-2,2}(\Omega))$. Combined with $u \in C(0, T; W_0^{2,2}(\Omega))$ this yields $uu_t \in C(0, T; L^1(\Omega))$ and, additionally, $u^2 \in C^1(0, T; L^1(\Omega))$. □

The following result bounds the growth of the norm of solutions.

Theorem 3.5. *If $u \in \mathcal{X}_T$ solves* (24) *then,*

$$\forall\, M, \epsilon > 0 \quad \exists\, \tau = \tau(M, \epsilon) > 0 \quad : \tag{35}$$
$$\Big(\|u_0\| < M\,, \quad t < \tau\Big) \implies \|u(t)\| < M + \epsilon\,.$$

A similar statement holds for the corresponding Navier problem.

Proof. We focus on the Dirichlet problem as the proof for the Navier case follows identically. We compute

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 = \langle \Delta u_t, \Delta u\rangle = \big(\Delta^2 u, u_t\big)_2 = \\ -\|\Delta^2 u\|_2^2 + \big(\Delta^2 u, \det(D^2 u)\big)_2 + \big(\Delta^2 u, \lambda f\big)_2 \le \\ -\|\Delta^2 u\|_2^2 + \|\Delta^2 u\|_2 \|\det(D^2 u)\|_2 + |\lambda|\, \|\Delta^2 u\|_2 \|f\|_2, \end{gathered}$$

by means of the application of the boundary conditions, the application of the equation and Hölder inequality. Young inequality leads to

$$\frac{d}{dt}\|u\|^2 \leq \|\det(D^2u)\|_2^2 + \lambda^2\|f\|_2^2;$$

now choosing $0 < \tau < T$ and integrating in time along the interval $(0,\tau)$ we find

$$\|u(\tau)\|^2 \leq \|u_0\|^2 + \int_0^\tau \|\det(D^2u)\|_2^2 + \lambda^2 \int_0^\tau \|f\|_2^2 <$$
$$M^2 + \int_0^\tau \|\det(D^2u)\|_2^2 + \lambda^2 \int_0^\tau \|f\|_2^2.$$

Arguing as in Step 2 of the proof of Theorem 3.2 we transform this inequality into

$$\|u(\tau)\|^2 < M^2 + C \max_{0\leq t\leq T} \|u\|^{5/2} \left(\int_0^T \|\Delta^2 u\|_2^2\right)^{3/4} \tau^{1/4} + \lambda^2 \int_0^\tau \|f\|_2^2.$$

Using the concavity of the square root we conclude

$$\|u(\tau)\| < M + C \max_{0\leq t\leq T} \|u\|^{5/4} \left(\int_0^T \|\Delta^2 u\|_2^2\right)^{3/8} \tau^{1/8} + |\lambda| \left(\int_0^\tau \|f\|_2^2\right)^{1/2}$$

and the statement follows by choosing a small enough τ. □

4. THE PARABOLIC PROBLEM WITHOUT SOURCE

In this section we consider the parabolic problem

$$\left\{ \begin{array}{rcll} u_t + \Delta^2 u &=& \det(D^2 u) & (x,t) \in \Omega \times (0,T), \\ u(x,0) &=& u_0(x) & x \in \Omega \\ u = u_\nu &=& 0 & (x,t) \in \partial\Omega \times (0,T). \end{array} \right. \tag{36}$$

4.1. Preliminary lemmas.

We start with the following result.

Lemma 4.1. *If $u = u(t)$ solves (36) then its energy*

$$J(u(t)) = \frac{1}{2}\int_\Omega |\Delta u(t)|^2 - \int_\Omega u_x(t) u_y(t) u_{xy}(t)$$

satisfies

$$\frac{d}{dt} J(u(t)) = -\int_\Omega u_t(t)^2 \leq 0\,.$$

Proof. Two integrations by parts show that

$$\frac{d}{dt}\int_\Omega |\Delta u|^2 = 2\langle \Delta u_t, \Delta u\rangle = 2\int_\Omega u_t \Delta^2 u\,.$$

Note that for any smooth function $v \in \mathcal{X}_T$,

$$\frac{d}{dt}\int_\Omega v_x v_y v_{xy} = \int_\Omega (v_{xt} v_y v_{xy} + v_x v_{yt} v_{xy} + v_x v_y v_{xyt})$$

and, since $v_t = 0$ on $\partial\Omega$, integrating by parts we obtain

$$\int_\Omega v_{xt} v_y v_{xy} = -\int_\Omega v_t \left(v_y v_{xy}\right)_x, \qquad \int_\Omega v_x v_{yt} v_{xy} = -\int_\Omega v_t \left(v_x v_{xy}\right)_y,$$

$$\int_\Omega v_x v_y v_{xyt} = \int_\Omega v_t \Big(v_x v_y\Big)_{xy} \,,$$

where, in the latter, we also used the condition that $|\nabla v| = 0$ on $\partial\Omega$. By collecting terms, this proves that

$$\frac{d}{dt}\int_\Omega v_x v_y v_{xy} = \int_\Omega \det(D^2 v)\, v_t \;.$$

By a density argument, the same holds true for the solution $u \in \mathcal{X}_T$ to (36). Hence,

$$\frac{d}{dt} J(u(t)) = \int_\Omega \Big(\Delta^2 u - \det(D^2 u)\Big)\, u_t = -\int_\Omega u_t^2 \,,$$

which proves the statement. □

Lemma 4.2. *Let $u_0 \in W_0^{2,2}(\Omega)$ be such that $J(u_0) < d$. Then:*
(i) if $u_0 \in \mathcal{N}_-$ the solution $u = u(t)$ to (36) *satisfies $J(u(t)) < d$ and $u(t) \in \mathcal{N}_-$ for all $t \in (0,T)$;*
(ii) if $u_0 \in \mathcal{N}_+$ the solution $u = u(t)$ to (36) *satisfies $J(u(t)) < d$ and $u(t) \in \mathcal{N}_+$ for all $t \in (0,T)$.*

Proof. If $J(u_0) < d$, then $J(u(t)) < d$ for all $t \in (0,T)$ in view of Lemma 4.1. Assume moreover that $u_0 \in \mathcal{N}_+$ and, for contradiction, that $u(t) \notin \mathcal{N}_+$ for some $t \in (0,T)$. Then, necessarily $u(t) \in \mathcal{N}$ for some $t \in (0,T)$ so that, by (13), $J(u(t)) \geq d$, contradiction. We may argue similarly if $u_0 \in \mathcal{N}_-$. □

Next, we prove a kind of L^2-*Cauchy property* for global solutions with bounded energy.

Lemma 4.3. *Let $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ be the corresponding solution to* (36). *Then*

$$\|u(t+\delta) - u(t)\|_2^2 \leq \delta\Big(J(u(t)) - J(u(t+\delta))\Big) \qquad \forall \delta > 0$$

and

$$\left(\frac{\|u(t+\delta)\|_2 - \|u(t)\|_2}{\delta}\right)^2 \leq \frac{J(u(t)) - J(u(t+\delta))}{\delta} \;. \tag{37}$$

In particular, the map $t \mapsto \|u(t)\|_2$ is differentiable and

$$\left(\frac{d}{dt}\|u(t)\|_2\right)^2 \leq -\frac{d}{dt} J(u(t)) \;.$$

Proof. By Hölder inequality, Fubini Theorem, and Lemma 4.1, we get

$$\begin{aligned} \|u(t+\delta) - u(t)\|_2^2 &= \int_\Omega \left|\int_t^{t+\delta} u_t(\tau)\right|^2 \leq \delta \int_\Omega \int_t^{t+\delta} u_t(\tau)^2 \\ &= \delta \int_t^{t+\delta} \Big(\int_\Omega u_t(\tau)^2\Big) = \delta\Big(J(u(t)) - J(u(t+\delta))\Big) \end{aligned}$$

which is the first inequality. By the triangle inequality and the just proved inequality we infer that

$$\Big(\|u(s+\delta)\|_2-\|u(s)\|_2\Big)^2\le\|u(s+\delta)-u(s)\|_2^2\le\delta\Big(J(u(t))-J(u(t+\delta))\Big)\qquad\forall\delta>0$$

which we may rewrite as (37). Finally, the estimate of the derivative follows by letting $\delta\to 0$. □

Also the derivative of the squared L^2-norm has an elegant form:

Lemma 4.4. *Let $u_0\in W_0^{2,2}(\Omega)$ and let $u=u(t)$ be the corresponding solution to* (36)*. Then for all $t\in[0,T)$ we have*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2+\|u(t)\|^2-3\int_\Omega u_x(t)u_y(t)u_{xy}(t)=0\,. \tag{38}$$

Proof. Multiply (36) by $u(t)$, integrate over Ω, and apply (8) to obtain (38). □

Finally, we prove that the nonlinear terms goes to the "correct" limit for $W_0^{2,2}(\Omega)$-bounded sequences.

Lemma 4.5. *Let $\{u_k\}$ be a bounded sequence in $W_0^{2,2}(\Omega)$. Then there exists $\overline{u}\in W_0^{2,2}(\Omega)$ such that $u_k\rightharpoonup\overline{u}$ in $W_0^{2,2}(\Omega)$ and*

$$\int_\Omega\phi\ \det(D^2u_k)\to\int_\Omega\phi\ \det(D^2\overline{u})\quad\forall\phi\in W_0^{2,2}(\Omega),$$

after passing to a suitable subsequence.

Proof. The first part is immediate and follows from the reflexivity of the Sobolev space $W_0^{2,2}(\Omega)$. The second part cannot be deduced in the same way because $L^1(\Omega)$ is not reflexive and consequently the sequence $\det(D^2u_k)$ could converge to a measure. For all $v,w\in C_0^\infty(\Omega)$ some integrations by parts show that

$$\int_\Omega w\ \det\left(D^2v\right)=\int_\Omega v_{x_1}v_{x_2}w_{x_1x_2}-\frac{1}{2}v_{x_2}^2w_{x_1x_1}-\frac{1}{2}v_{x_1}^2w_{x_2x_2}. \tag{39}$$

A density argument shows that the same is true for all $v,w\in W_0^{2,2}(\Omega)$. Therefore for any $\phi\in W_0^{2,2}(\Omega)$ and any k we have

$$\int_\Omega\phi\ \det\left(D^2u_k\right)=\int_\Omega(u_k)_{x_1}(u_k)_{x_2}\phi_{x_1x_2}-\frac{1}{2}(u_k)_{x_2}^2\phi_{x_1x_1}-\frac{1}{2}(u_k)_{x_1}^2\phi_{x_2x_2}.$$

By compact embedding we know that $u_k\to u$ strongly in $W_0^{1,4}(\Omega)$ since $u_k\rightharpoonup u$ weakly in $W_0^{2,2}(\Omega)$, and thus

$$\lim_{k\to\infty}\int_\Omega\phi\ \det\left(D^2u_k\right)=\int_\Omega\overline{u}_{x_1}\overline{u}_{x_2}\phi_{x_1x_2}-\frac{1}{2}\overline{u}_{x_2}^2\phi_{x_1x_1}-\frac{1}{2}\overline{u}_{x_1}^2\phi_{x_2x_2},$$

after passing to a suitable subsequence. Applying again (39) leads to

$$\lim_{k\to\infty}\int_\Omega\phi\ \det\left(D^2u_k\right)=\int_\Omega\phi\ \det\left(D^2\overline{u}\right),$$

after passing to a suitable subsequence. □

4.2. **Finite time blow-up.** Our first result proves the existence of solutions to (36) which blow up in finite time.

Theorem 4.6. *Let $u_0 \in \mathcal{N}_-$ be such that $J(u_0) \le d$. Then the solution $u = u(t)$ to* (36) *blows up in finite time, that is, there exists $T > 0$ such that $\|u(t)\| \to +\infty$ as $t \nearrow T$. Moreover, the blow up also occurs in the $W_0^{1,4}(\Omega)$-norm, that is, $\|u(t)\|_{W_0^{1,4}(\Omega)} \to +\infty$ as $t \nearrow T$.*

Proof. Again, since $u_0 \notin \mathcal{N}$, we know that, by Lemma 4.1, we have $J(u(t)) < d$ for all $t > 0$. Therefore, possibly by translating t, we may assume that $J(u(0)) < d$ and, from now on, we rename $u_0 = u(0)$. We use here a refinement of the concavity method by Levine [24], see also [30, 35]. Assume for contradiction that the solution $u = u(t)$ to (36) is global and define

$$M(t) := \frac{1}{2}\int_0^t \|u(s)\|_2^2$$

so that, by Theorem 3.2 and Corollary 3.4, $M \in C^2(0,\infty)$. Then

$$M'(t) = \frac{\|u(t)\|_2^2}{2}$$

and, by (38),

$$M''(t) = -3J(u(t)) + \frac{\|u(t)\|^2}{2} .$$

By the assumptions on u_0 and by Lemma 4.2 we know that $u(t) \in N_-$ for all $t \ge 0$. In turn, by Theorem 2.5, we infer that $\|u(t)\|^2 > 6d$ for all $t \ge 0$. Hence, recalling Lemma 4.1 and the assumptions, we get

$$M''(t) \ge -3J(u_0) + \frac{\|u(t)\|^2}{2} > 3(d - J(u_0)) > 0 \qquad \text{for all } t \ge 0 .$$

This shows that

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} M'(t) = +\infty . \tag{40}$$

By Lemma 4.1 we also infer that

$$J(u(t)) = J(u_0) - \int_0^t \|u_t(s)\|_2^2$$

so that

$$M''(t) = 3\int_0^t \|u_t(s)\|_2^2 - 3J(u_0) + \frac{\|u(t)\|^2}{2} > 3\int_0^t \|u_t(s)\|_2^2$$

since $\|u(t)\|^2 > 6d > 6J(u_0)$. By multiplying the previous inequality by $M(t) > 0$ and by using Hölder inequality, we get

$$\begin{aligned} M''(t)M(t) &\ge \frac{3}{2}\int_0^t \|u_t(s)\|_2^2 \int_0^t \|u(s)\|_2^2 \ge \\ \frac{3}{2}\Big(\int_0^t\int_\Omega u(s)u_t(s)\Big)^2 &= \frac{3}{2}\Big(M'(t) - M'(0)\Big)^2 . \end{aligned}$$

By (40) we know that there exists $\tau > 0$ such that $M'(t) > 7M'(0)$ for $t > \tau$ so that the latter inequality becomes

$$M''(t)M(t) > \frac{54}{49}M'(t)^2 \qquad \text{for all } t > \tau \, . \tag{41}$$

This shows that the map $t \mapsto M(t)^{-5/49}$ has negative second derivative and is therefore concave on $[\tau, +\infty)$. Since $M(t)^{-5/49} \to 0$ as $t \to \infty$ in view of (40), we reach a contradiction. This shows that the solution $u(t)$ is not global and, by Theorem 3.2, that there exists $T > 0$ such that $\|u(t)\| \to +\infty$ as $t \nearrow T$.

Since by Lemma 4.2 we have that $u(t) \in \mathcal{N}_-$ for all $t \geq 0$, by (17) we infer that

$$\|u(t)\|^2 < 3\int_\Omega u_x(t)u_y(t)u_{xy}(t) \leq \frac{3}{4}\|u(t)\| \, \|u(t)\|^2_{W^{1,4}_0(\Omega)} \qquad \text{for all } t \geq 0$$

so that $\|u(t)\| < \frac{3}{4}\|u(t)\|^2_{W^{1,4}_0(\Omega)}$ and the $W^{1,4}_0(\Omega)$-norm blows up as $t \nearrow T$. □

Next, we state a blow up result without assuming that the initial energy $J(u_0)$ is smaller than the mountain pass level d. Let λ_1 denote the least Dirichlet eigenvalue of the biharmonic operator in Ω and assume that $u_0 \in W^{2,2}_0(\Omega)$ satisfies

$$\lambda_1\|u_0\|_2^2 > 6J(u_0) \, . \tag{42}$$

By Poincaré inequality $\|u_0\|^2 \geq \lambda_1\|u_0\|_2^2$, we see that if u_0 satisfies (42), then $u_0 \in \mathcal{N}_-$. However, the energy $J(u_0)$ may be larger than d. For instance, let e^1 denote an eigenfunction corresponding to λ_1 with the sign implying $\int_\Omega e^1_x e^1_y e^1_{xy} > 0$. If we take $u_0 = \alpha e^1$, then (42) will be satisfied for any $\alpha > \overline{\alpha}$ where $\overline{\alpha}$ is the unique value of $\alpha > 0$ such that $\overline{\alpha}e^1 \in \mathcal{N}$. And, by (13), we know that $J(\overline{\alpha}e^1) > d$. So, for $\alpha > \overline{\alpha}$ sufficiently close to $\overline{\alpha}$ we have $J(\alpha e^1) > d$, that is, we are above the mountain pass level.

Assumption (42) yields finite time blow-up.

Theorem 4.7. *Assume that $u_0 \in W^{2,2}_0(\Omega)$ satisfies* (42)*. Then the solution $u = u(t)$ to* (36) *blows up in finite time, that is, there exists $T > 0$ such that $\|u(t)\| \to +\infty$ and $\|u(t)\|_{W^{1,4}_0(\Omega)} \to +\infty$ as $t \nearrow T$.*

Proof. We first claim that if $u = u(t)$ is a global solution to (36) then

$$\liminf_{t\to\infty} \|u(t)\| < +\infty \, . \tag{43}$$

For contradiction, assume that the solution $u = u(t)$ to (36) is global and that

$$\|u(t)\| \to +\infty \qquad \text{as } t \to +\infty \, . \tag{44}$$

In what follows, we use the same tools as in the proof of Theorem 4.6. Consider again

$$M(t) := \frac{1}{2}\int_0^t \|u(s)\|_2^2 \, .$$

Then

$$M''(t) = -3J(u(t)) + \frac{\|u(t)\|^2}{2} \to +\infty \qquad \text{as } t \to +\infty$$

because of (44) and Lemma 4.1 (the map $t \mapsto -3J(u(t))$ is increasing). This proves again (40).

By Lemma 4.1 and using (44) we also infer that there exists $\tau > 0$ such that

$$M''(t) > 3\int_0^t \|u_t(s)\|_2^2 \qquad \forall t > \tau\ .$$

By multiplying the previous inequality by $M(t) > 0$ and by using Hölder inequality, we find

$$M''(t)M(t) \geq \frac{3}{2}\Big(M'(t) - M'(0)\Big)^2 \qquad \forall t > \tau$$

and that (41) holds, for a possibly larger τ. The same concavity argument used in the proof of Theorem 4.6 leads to a contradiction. Hence, (44) cannot occur and (43) follows.

Next, by Poincaré inequality and Lemma 4.1, (38) yields

$$\frac{d}{dt}\|u(t)\|_2^2 = -6J(u(t)) + \|u(t)\|^2 \geq -6J(u_0) + \lambda_1\|u(t)\|_2^2\ .$$

By putting $\psi_0(t) := -6J(u_0) + \lambda_1\|u(t)\|_2^2$, the previous inequality reads $\psi_0'(t) \geq \lambda_1\psi_0(t)$. Since (42) yields $\psi_0(0) > 0$, this proves that $\psi_0(t) \to \infty$ as $t \to \infty$. Hence, by invoking again Poincaré inequality, we see that also (44) holds, a situation that we ruled out by proving (43). This contradiction shows that $T < \infty$. The blow up of the $W_0^{1,4}(\Omega)$-norm follows as in the proof of Theorem 4.6. □

Let $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ be the local solution to (36). According to Theorem 3.2, the solution blows up at some $T > 0$ if

(45) $$\lim_{t\to T}\ \|u(t)\|\ =\ +\infty\ .$$

We wish to investigate if the (finite time) blow up also occurs in different ways. In particular, we wish to analyze the following forms of blow up:

(46) $$\lim_{t\to T}\ \|u\|_{L^2(0,t;W_0^{2,2}(\Omega))}\ =\ +\infty\ ,$$

(47) $$\lim_{t\to T}\ \|u(t)\|_2\ =\ +\infty\ ,$$

(48) $$\lim_{t\to T}\ \|u\|_{L^4(0,t;W_0^{1,4}(\Omega))}\ =\ +\infty\ .$$

Clearly, (47) implies (45). We show that also further implications hold true.

Theorem 4.8. *Let $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ be the local solution to* (36). *Assume that* (45) *occurs for some finite $T > 0$. Then there exists $\tau \in (0,T)$ such that $u(t) \in \mathcal{N}_-$ for all $t > \tau$.*

Moreover:

(i) If (46) *occurs, then* (47) *occurs.*

(ii) If (47) *occurs, then* (48) *occurs.*

Finally, (47) *occurs if and only if*

(49) $$\lim_{t\to T}\int_0^t\left(\int_\Omega \Delta u(s)|\nabla u(s)|^2\right)\ =\ -\infty\ .$$

Proof. For contradiction, assume that there exists a sequence $t_n \to T$ such that $u(t_n) \in (\mathcal{N} \cup \mathcal{N}_+)$. Then

$$\|u(t_n)\|^2 \geq 3 \int_\Omega u_x(t_n) u_y(t_n) u_{xy}(t_n) \qquad \forall n$$

which, in view of (45), implies that

$$J(u(t_n)) = \frac{1}{2}\|u(t_n)\|^2 - \int_\Omega u_x(t_n) u_y(t_n) u_{xy}(t_n) \geq \frac{1}{6}\|u(t_n)\|^2 \to +\infty$$

as $n \to \infty$. This contradicts Lemma 4.1. Hence, there exists $\tau \in (0,T)$ such that $u(t) \in \mathcal{N}_-$ for all $t > \tau$.

Integrating (38) over $[0,t]$ for $0 < t < T$ yields

$$\|u(t)\|_2^2 = \|u_0\|_2^2 + \int_0^t \left(-2\|u(s)\|^2 + 6 \int_\Omega u_x(s) u_y(s) u_{xy}(s) \right) . \tag{50}$$

By Lemma 4.1 we know that $J(u(t)) \leq J(u_0)$, that is,

$$2 \int_\Omega u_x(t) u_y(t) u_{xy}(t) \geq \|u(t)\|^2 - 2J(u_0) \qquad \forall t \in (0,T) .$$

Hence, (50) yields

$$\|u(t)\|_2^2 \geq \|u_0\|_2^2 + \int_0^t \|u(s)\|^2 - 6J(u_0)\, t \qquad \forall t \in (0,T) .$$

Letting $t \to T$ we see that (46) implies (47).

Using (17) into (50) yields

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \int_0^t \left(-2\|u(s)\|^2 + \frac{3}{2}\|u(s)\|\, \|u(s)\|_{W_0^{1,4}(\Omega)}^2 \right) . \tag{51}$$

By the Young inequality $\frac{3}{2}ab \leq 2a^2 + \frac{9}{32}b^2$, (51) becomes

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \frac{9}{32}\|u\|_{L^4(0,t;W_0^{1,4}(\Omega))}^4 .$$

Letting $t \to T$, this proves that if (47) occurs, then also (48) occurs.

Assume now that (49) occurs and, using (16), rewrite (50) as

$$\|u(t)\|_2^2 = \|u_0\|_2^2 + \int_0^t \left(-2\|u(s)\|^2 - \frac{3}{2} \int_\Omega \Delta u(s)\, |\nabla u(s)|^2 \right) . \tag{52}$$

Two cases may occur. If (46) holds, then by the just proved statement (i), (47) occurs. If (46) does not hold, so that $\|u\|_{L^2(0,t;W_0^{2,2}(\Omega))}$ remains bounded, then (52) shows again that (47) occurs. Therefore, in any case, if (49) occurs, then (47) occurs.

Finally, from (52) we see that

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 - \frac{3}{2} \int_0^t \left(\int_\Omega \Delta u(s)\, |\nabla u(s)|^2 \right)$$

which proves that (47) implies (49). □

4.3. **Global solutions.** For suitable initial data, not only the solution is global but it vanishes in infinite time.

Theorem 4.9. *Let $u_0 \in \mathcal{N}_+$ be such that $J(u_0) \leq d$. Then the solution $u = u(t)$ to* (36) *is global and $u(t) \to 0$ in $W^{4,2}(\Omega)$ as $t \to +\infty$.*

Proof. Since $u_0 \notin \mathcal{N}$, we know that it is not a stationary solution to (36), that is, it does not solve (9). Hence, by Lemma 4.1 we have $J(u(t)) < d$ for all $t > 0$. By Lemma 4.2 and Theorem 2.5 we infer that $u(t)$ remains bounded in $W_0^{2,2}(\Omega)$ so that, by Theorem 4.6, the solution is global. If $\|u_t\|_2 \geq c > 0$ for all $t > 0$, then by Lemma 4.1 we would get $J(u(t)) \to -\infty$ as $t \to \infty$ against $u(t) \in N_+$, see again Lemma 4.2. Hence, $u_t(t) \to 0$ in $L^2(\Omega)$, on a suitable sequence.

Moreover, the boundedness of $\|u(t)\|$ implies that there exists $\overline{u} \in W_0^{2,2}(\Omega)$ such that $u(t) \rightharpoonup \overline{u}$ in $W_0^{2,2}(\Omega)$ as $t \to \infty$ on the sequence. Note also that, by Lemma 4.5, for all $\phi \in W_0^{2,2}(\Omega)$ we have

$$\int_\Omega \phi \det(D^2 u(t)) \to \int_\Omega \phi \det(D^2 \overline{u}).$$

Therefore, if we test (36) with some $\phi \in W_0^{2,2}(\Omega)$, and we let $t \to \infty$ on the above found sequence, we get

$$0 = \int_\Omega u_t(t)\phi + \int_\Omega \Delta u(t)\Delta\phi - \int_\Omega \det(D^2 u(t))\phi \to \int_\Omega \Delta\overline{u}\Delta\phi - \det(D^2\overline{u})\phi$$

which shows that $\overline{u}$ solves (9). Since the only solution to (9) at energy level below d is the trivial one, we infer that $\overline{u} = 0$. Writing (36) as

$$\Delta^2 u(t) = -u_t(t) + \det(D^2 u(t))$$

we see that $\Delta^2 u(t)$ is uniformly bounded in $L^1(\Omega)$. Whence, by arguing as in the proof of Theorem 2.3, we first infer that $\Delta^2 u(t)$ is bounded in $W^{-s,2}(\Omega)$ for all $s > 1$ and then, by a bootstrap argument, that

$$\Delta^2 u(t) = -u_t(t) + \det(D^2 u(t)) \to 0 \text{ strongly in } L^2(\Omega)$$

so that $u(t) \to 0$ in $W^{4,2}(\Omega)$ on the sequence.

By Lemma 4.1, we infer that $J(u(t)) \to 0$ regardless of how $t \to \infty$. Since $u(t) \in \mathcal{N}_+$ for all $t \geq 0$, we also have that $J(u(t)) \geq \|u(t)\|^2/6$ for all t. These facts enable us to conclude that all the above convergences occur as $t \to \infty$, not only on some subsequence. □

Theorems 4.6 and 4.7 determine a wide class of initial data $u_0 \in W_0^{2,2}(\Omega)$ which ensure that the solution to (36) blows up in finite time. One can wonder whether the blow up might also occur in infinite time. This happens, for instance, in semilinear second order parabolic equations at critical growth, see [28, 32]. If $T = +\infty$, we denote by

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{u(s) \,:\, s \geq t\}}$$

the ω-limit set of $u_0 \in W_0^{2,2}(\Omega)$, where the closure is taken in $W_0^{2,2}(\Omega)$. We show here that infinite time blow up cannot occur for the fourth order parabolic equation (36).

Theorem 4.10. *Let $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ be the local solution to* (36). *If $T = +\infty$ then the ω-limit set $\omega(u_0)$ is a nonempty bounded connected subset of $W_0^{2,2}(\Omega)$ which consists of solutions to* (9). *In particular, this means that there exists a solution $\overline{u}$ to* (9) *such that $u(t) \to \overline{u}$ in $W_0^{2,2}(\Omega)$ up to a subsequence and, if $\overline{u}$ is an isolated solution to* (9)*, then $u(t) \to \overline{u}$ in $W_0^{2,2}(\Omega)$ as $t \to \infty$ (without passing on a subsequence). These convergences are, in fact, also in $W^{4,2}(\Omega)$.*

Proof. If $u = u(t)$ is a global solution to (36), then we know that (43) holds. We claim that if

$$C := \liminf_{t\to\infty} \|u(t)\| < \limsup_{t\to\infty} \|u(t)\| = +\infty \,, \tag{53}$$

then $J(u(t)) \geq d$ for all $t \geq 0$ and $C > 0$. By Lemma 4.1, the map $t \mapsto J(u(t))$ admits a limit as $t \to \infty$. If this limit were smaller than d (including $-\infty$), then we would have $J(u(\overline{t})) < d$ for some $\overline{t} > 0$. By (13) this implies that either $u(\overline{t}) \in \mathcal{N}_+$ or $u(\overline{t}) \in \mathcal{N}_-$. In the first situation, Theorem 4.9 implies that $\|u(t)\|_{W^{4,2}(\Omega)} \to 0$ as $t \to \infty$. In the second situation, Theorem 4.6 implies that $\|u(t)\| \to \infty$ in finite time. In both cases we contradict (53). Hence, if (53) holds then

$$J(u(t)) \geq \Upsilon := \lim_{t\to\infty} J(u(t)) \geq d \,. \tag{54}$$

If $C = 0$ in (53), then there exists a divergent sequence $\{t_m\}$ such that $\|u(t_m)\| \to 0$ so that $J(u(t_m)) \to 0$, contradicting (54). By Lemma 4.1 we know that

$$\int_0^\infty \|u_t(t)\|_2^2 = J(u_0) - \Upsilon \tag{55}$$

so that $u_t \in L^2(\mathbb{R}_+; L^2(\Omega))$ and

$$\liminf_{t\to\infty} \; \|u_t(t)\|_2 \; = \; 0 \,. \tag{56}$$

We claim that also

$$u \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \,. \tag{57}$$

If $u \in L^\infty(\mathbb{R}_+; W_0^{2,2}(\Omega))$, then the statement follows directly from Poincaré inequality. So, assume that $t \mapsto \|u(t)\|$ is not bounded in $\mathbb{R}_+$ so that, by (43), we know that necessarily (53) holds. Let $\Lambda := \max\{2C, 8J(u_0)\} > 0$ and consider the two sets

$$\Theta_- := \{t \geq 0;\ \|u(t)\|^2 \leq \Lambda\} \,, \qquad \Theta_+ := \{t \geq 0;\ \|u(t)\|^2 > \Lambda\} \,.$$

We have $\Theta_- \cup \Theta_+ = \mathbb{R}_+$ and, in view of (53), both $\Theta_+ \neq \emptyset$ and $\Theta_- \neq \emptyset$. Note that for $t \in \Theta_+$ we have $\|u(t)\|^2 > 8J(u_0) \geq 8J(u(t))$ in view of Lemma 4.1 so that $u(t) \in \mathcal{N}_-$ and, by (38), the map $t \mapsto \|u(t)\|_2$ is strictly increasing in Θ_+. By (53) we know that t changes infinitely many times between Θ_+ and Θ_-. As long as $t \in \Theta_-$, by Poincaré inequality we have $\lambda_1 \|u(t)\|_2^2 \leq \|u(t)\|^2 \leq \Lambda$ and therefore $\|u(t)\|_2$ remains uniformly bounded. Moreover, by the just proved monotonicity, as long as $t \in \Theta_+$ we know that $\|u(t)\|_2^2 \le \|u(\bar{t})\|_2^2 \le \Lambda/\lambda_1$ where $\bar{t}$ is the first instant where t exists Θ_+. This proves (57).

Next, note that if c denotes positive constants which may vary from line to line, we may rewrite (38) as

$$\begin{aligned}\|u(t)\|^2 &= 6J(u(t)) + 2\int_\Omega u(t)u_t(t) \le 6J(u_0) + 2\int_\Omega |u(t)u_t(t)| \\ &\le c\Big(1 + \|u(t)\|_2\,\|u_t(t)\|_2\Big) \le c\Big(1 + \|u_t(t)\|_2\Big)\end{aligned}$$

where we also used Lemma 4.1 (first inequality), Hölder inequality (second inequality), and (57) (third inequality). By squaring, we obtain

$$\|u(t)\|^4 \le c_1 + c_2\|u_t(t)\|_2^2\ . \tag{58}$$

Put $\Gamma_t := \{s \ge t;\ \|u(s)\|^4 \ge c_1 + 1\}$ where c_1 is as in (58) and let $|\Gamma_t|$ denote the measure of Γ_t. Then $|\Gamma_t| \to 0$ as $t \to \infty$ because of (55) and (58). Take $M := \sqrt[4]{c_1+1}$, $\epsilon = 1$, and let $\tau > 0$ be the number given by (35). Then take t sufficiently large so that $|\Lambda_t| < \tau$. By (35), for any such t we have $\|u(t)\| < M+1$, which proves that

$$u \in L^\infty(\mathbb{R}_+; W_0^{2,2}(\Omega))\ . \tag{59}$$

By (56) there exists a diverging sequence $\{t_k\}$ such that $u_t(t_k) \to 0$ in $L^2(\Omega)$ as $k \to \infty$. By (59), up to a further subsequence, we have $u(t_k) \rightharpoonup \overline{u}$ in $W_0^{2,2}(\Omega)$ for some $\overline{u} \in W_0^{2,2}(\Omega)$. By testing (36) with $\varphi \in C_0^\infty(\Omega)$ and letting $k \to \infty$, we see that $\overline{u}$ solves (9). We may apply these arguments to several subsequences; hence, due to the continuity of the map $t \mapsto \|u(t)\|$, the ω-limit set $\omega(u_0)$ is a nonempty connected subset of $W_0^{2,2}(\Omega)$ which consists of solutions to (9). Finally, the convergences may be improved to $W^{4,2}(\Omega)$ by arguing as in Theorem 4.9. □

Remark 4.11. In Theorem 4.10, by "$\overline{u}$ is an isolated solution", we mean that there exists a $W_0^{2,2}(\Omega)$-neighborhood of $\overline{u}$ which contains no further solutions to (9). In general, Theorem 4.10 cannot be improved with the statement that the *whole* trajectory converges, see [31, 32] and references therein for second order equations. Note also that from Lemma 4.3 and (54) we infer that if u is a global solution, then

$$\lim_{t\to\infty} \|u(t+\delta) - u(t)\|_2 = 0 \qquad \forall \delta > 0.$$

This shows that the convergence to $\omega(u_0)$ occurs "slowly".

Finally, we prove a squeezing property which is typical of dissipative dynamical systems. Since (36) is indeed dissipative when dealing with global solutions, we restrict our attention to this case. Consider the sequence of Dirichlet eigenvalues $\{\lambda_m\}$ of the biharmonic operator and denote by $\{e^m\}$ the sequence of corresponding $W_0^{2,2}(\Omega)$-normalized orthogonal eigenfunctions. It is well-known that

$$v = \sum_{m=1}^{\infty} \left(\int_\Omega \Delta v \Delta e^m\right) e^m \qquad \forall v \in W_0^{2,2}(\Omega)$$

where the series converges in the $W_0^{2,2}(\Omega)$-norm. For all $k \geq 2$ denote by P_k the projector onto the space H_k spanned by $\{e^1, ..., e^{k-1}\}$ so that

$$P_k v = \sum_{m=1}^{k-1} \left(\int_\Omega \Delta v \Delta e^m \right) e^m \qquad \forall v \in W_0^{2,2}(\Omega) \, .$$

Finally, we recall the improved Poincaré inequality

$$\lambda_k \|v\|_2^2 \leq \|v\|^2 \qquad \forall v \in H_k^\perp \tag{60}$$

where $H_k^\perp$ denotes the orthogonal complement of H_k, namely the closure of the infinite dimensional space spanned by $\{e^k, e^{k+1}, ...\}$. Roughly speaking, the next result states that the asymptotic behavior of the solutions to (36) is determined by a finite number of modes.

Theorem 4.12. *Let $u = u(t)$ and $v = v(t)$ be the solutions to* (36) *corresponding to initial data $u_0 \in W_0^{2,2}(\Omega)$ and $v_0 \in W_0^{2,2}(\Omega)$, respectively. Assume that u and v are global solutions to* (36)*. There exists $k \in \mathbb{N}$, depending only on $\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}$ and $\|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}$, such that if $P_k u(t) = P_k v(t)$ for all $t \geq 0$, then*

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{W^{s,2}(\Omega)} = 0 \qquad \text{for all } s \in [0, 2) \, .$$

Proof. Since $u, v \in C(\mathbb{R}_+; W_0^{2,2}(\Omega))$, by Theorem 4.10 we know that

$$u, v \in L^\infty(\mathbb{R}_+; W_0^{2,2}(\Omega)) \, . \tag{61}$$

We first claim that there exists $\mu > 0$ such that for all $u, v \in W_0^{2,2}(\Omega)$ we have

$$\int_\Omega \Big(\det(D^2u) - \det(D^2v)\Big)(u - v) \leq \mu\Big(\|u\| + \|v\|\Big)\|u - v\| \, \|u - v\|_\infty \, . \tag{62}$$

To see this, let us rewrite

$$\det(D^2u) - \det(D^2v) =$$
$$u_{xx}(u_{yy} - v_{yy}) + v_{yy}(u_{xx} - v_{xx}) + u_{xy}(v_{xy} - u_{xy}) + v_{xy}(v_{xy} - u_{xy})$$

so that, by Hölder inequality,

$$\begin{aligned} \|\det(D^2u) - \det(D^2v)\|_1 &\leq \|u_{xx}\|_2 \, \|u_{yy} - v_{yy}\|_2 + \|v_{yy}\|_2 \, \|u_{xx} - v_{xx}\|_2 \\ &\quad + \|u_{xy}\|_2 \, \|v_{xy} - u_{xy}\|_2 + \|v_{xy}\|_2 \, \|v_{xy} - u_{xy}\|_2 \\ &\leq \mu\Big(\|u\| + \|v\|\Big)\|u - v\| \, . \end{aligned}$$

Hence, by applying once more, Hölder inequality we obtain

$$\begin{aligned} \int_\Omega \Big(\det(D^2u) - \det(D^2v)\Big)(u - v) &\leq \|\det(D^2u) - \det(D^2v)\|_1 \, \|u - v\|_\infty \\ &\leq \mu\Big(\|u\| + \|v\|\Big)\|u - v\| \, \|u - v\|_\infty \, , \end{aligned}$$

which proves (62).

By subtracting the two equations relative to u and v we obtain

$$w_t + \Delta^2 w = \det(D^2u) - \det(D^2v) \tag{63}$$

where $w(t) = u(t) - v(t)$. Multiply (63) by w and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|w(t)\|^2 = \int_\Omega \Big(\det(D^2u(t)) - \det(D^2v(t))\Big)w(t)\ .$$

By (62) the latter may be estimated as

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|w(t)\|^2 \le \mu\Big(\|u(t)\| + \|v(t)\|\Big)\|w(t)\|\,\|w(t)\|_\infty\ .$$

In turn, by the Gagliardo-Nirenberg inequality $\|w\|_\infty \le c\|w\|^{1/2}\,\|w\|_2^{1/2}$ (see [13, 29]) we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|w(t)\|^2 \le$$

$$\mu\Big(\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))} + \|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}\Big)\|w(t)\|^{3/2}\,\|w(t)\|_2^{1/2}\ .$$

By recalling the assumption that $P_kw(t) = 0$, that is $w(t) \in H_k^\perp$, and by using (60) we then get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2\\
&\le \Big[\mu\Big[\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))} + \|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}\Big]\|w(t)\|_2^{1/2} - \|w(t)\|^{1/2}\Big]\,\|w(t)\|^{3/2}\\
&\quad \le \Big[\mu\Big[\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))} + \|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}\Big] - \lambda_k^{1/4}\Big]\|w(t)\|_2^{1/2}\|w(t)\|^{3/2}.
\end{aligned}$$

Take k large enough so that $\lambda_k > \mu^4\Big(\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))} + \|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}\Big)^4$ and put

$$\omega_k := \lambda_k^{1/4} - \mu\Big(\|u\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))} + \|v\|_{L^\infty(\mathbb{R}_+;W_0^{2,2}(\Omega))}\Big) > 0\ ,$$

$$\delta_k := 2\omega_k\lambda_k^{3/4} > 0\ .$$

By (60) we may finally rewrite the last inequality as

$$\frac{d}{dt}\|w(t)\|_2^2 \le -2\omega_k\|w(t)\|_2^{1/2}\|w(t)\|^{3/2} \le -\delta_k\|w(t)\|_2^2$$

which, upon integration, gives

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2\,e^{-\delta_k t} \qquad \forall t \ge 0 \tag{64}$$

and the statement follows for $s = 0$ by letting $t \to \infty$.

By interpolation, we know that

$$\|u(t) - v(t)\|_{W^{s,2}(\Omega)}^2 \le \|u(t) - v(t)\|_2^{2-s}\|u(t) - v(t)\|^s \qquad \text{for all } s \in (0, 2)\,;$$

the statement follows for all such s by combining (61) and (64). □

5. Further results and open problems

Monotonicity of the L^2-norm. It is clear that the limits in (46) and in (48) do exist due to the fact that they involve increasing functions of t. Less obvious is the existence of the limit in (47). The next result gives some monotonicity properties of the map $t \mapsto \|u(t)\|_2$ which guarantee that also the limit in (47) exists.

Proposition 5.1. *Let λ_1 denote the least eigenvalue of the biharmonic operator under Dirichlet boundary conditions in Ω. Take $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ denote the corresponding local solution to* (36)*.*

(i) If (42) *holds, then the map $t \mapsto \|u(t)\|_2$ is strictly increasing on $[0, T)$.*

(ii) If $u_0 \in \mathcal{N}_-$ and $J(u_0) < d$, then the map $t \mapsto \|u(t)\|_2$ is strictly increasing on $[0, T)$.

(iii) If $u_0 \in \mathcal{N}_+$ and $J(u_0) < d$, then the map $t \mapsto \|u(t)\|_2$ is strictly decreasing on $[0, T)$.

Moreover, the map $t \mapsto \|u(t)\|_2$ is strictly increasing (resp. decreasing) whenever $u(t) \in \mathcal{N}_-$ (resp. $u(t) \in \mathcal{N}_+$).

Finally, the map $t \mapsto \|u(t)\|_2$ is differentiable and

$$\left(\frac{d}{dt}\|u(t)\|_2\right)^2 \leq -\frac{d}{dt}J(u(t)) \, .$$

Proof. In view of the definition of $\mathcal{N}_\pm$ and Lemma 4.2, (38) proves directly statements (ii) and (iii) and the corresponding strict monotonicity of the map $t \mapsto \|u(t)\|_2$ whenever $u(t) \in \mathcal{N}_\pm$.

On the other hand, by Poincaré inequality, (38) yields

$$\frac{d}{dt}\|u(t)\|_2^2 = -6J(u(t)) + \|u(t)\|^2 \geq -6J(u(t)) + \lambda_1\|u(t)\|_2^2 =: \psi(t) \, .$$

By the assumption in (i) we infer that $\psi(0) > 0$ so that the map $t \mapsto \|u(t)\|_2$ is initially strictly increasing, say on some maximal interval $(0, \delta)$ where $\delta > 0$ is the first time where $\psi(\delta) = 0$. If such δ exists then, by Lemma 4.1, also $t \mapsto \psi(t)$ is strictly increasing on $(0, \delta)$ so that $\psi(\delta) > \psi(0) > 0$, contradiction. Therefore δ does not exist and the maximal interval of strict monotonicity for $t \mapsto \|u(t)\|_2$ coincides with $(0, T)$.

Finally, the differentiability of the map $t \mapsto \|u(t)\|_2$ and the estimate of its derivative follows from Lemma 4.3. □

Finite time blow-up for Navier boundary conditions. Consider the initial-boundary value problem

$$\begin{cases} u_t = \det(D^2u) - \Delta^2 u & \text{in } \Omega \times (0, T), \\ u(x,t) = \Delta u(x,t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{65}$$

We will prove that the solution to it blows up in finite time provided u_0 is large enough in a sense to be specified in the following. For simplicity we focus on the radial problem set on the unit ball, $\Omega = B_1(0)$, so problem (65) simplifies to

$$\begin{cases} u_t = \frac{u_r u_{rr}}{r} - \Delta_r^2 u & \text{for } r \in [0,1)\,,\ t > 0, \\ u(1,t) = \Delta_r u(1,t) = 0 & \text{for } t > 0, \\ u(r,0) = u_0(r) & \text{for } r \in [0,1) \end{cases} \tag{66}$$

where $u = u(r,t)$ and $\Delta_r(\cdot) = \frac{1}{r}[r(\cdot)_r]_r$ is the radial Laplacian. Note that smoothness of the solution implies the symmetry condition $u_r(0,t) = 0$ for all $t \geq 0$ during the lapse of existence.

Theorem 5.2. *Let $u = u(r,t)$ be a smooth solution to* (66). *If*

$$\int_0^1 \left(\frac{4}{5} r^5 - \frac{9}{4} r^4 + \frac{5}{2} r^2 \right) (u_0)_r \, dr$$

is large enough, then there exists a $T^* < \infty$ such that u ceases to exist when $t \to T^*$.

Proof. We begin our proof with the following identity

$$\int_0^1 \left(\frac{4}{5} r^5 - \frac{9}{4} r^4 + \frac{5}{2} r^2 \right) u_r \, dr = - \int_0^1 (4r^4 - 9r^3 + 5r) u \, dr,$$

where the integration by parts made use of the boundary condition $u(1,t) = 0$ and the fact that one of the roots of the polynomial inside the left hand side integral is located at the origin. Now, using equation (66) we get

$$\begin{aligned} -\frac{d}{dt} \int_0^1 (4r^4 - 9r^3 + 5r) u \, dr &= - \int_0^1 (4r^3 - 9r^2 + 5) u_r u_{rr} \, dr \\ &\quad + \int_0^1 (4r^3 - 9r^2 + 5) [r(\Delta_r u)_r]_r \, dr. \end{aligned}$$

The first integral on the right hand side can be estimated integrating by parts

$$- \int_0^1 (4r^3 - 9r^2 + 5) u_r u_{rr} \, dr = \int_0^1 (6r - 9)(u_r)^2 \, r \, dr,$$

where we have used the symmetry condition $u_r(0,t) = 0$ and the fact that the polynomial inside the integral on the left hand side has one root at $r = 1$. Now we estimate the integral

$$\begin{aligned} \int_0^1 (4r^3 - 9r^2 + 5) [r(\Delta_r u)_r]_r \, dr &= - \int_0^1 (12r^2 - 18r)(\Delta_r u)_r \, r \, dr \\ &= \int_0^1 (36r^2 - 36r) \Delta_r u \, dr \\ &= -36 \int_0^1 u_r \, r \, dr, \end{aligned}$$

where the boundary terms vanish due to the presence of roots of the polynomial at the boundary points in the first case, due to the root of the polynomial at the origin and the boundary condition $\Delta_r u(1,t) = 0$ in the second case and due to the roots of the polynomial and the symmetry condition $u_r(0,t) = 0$ in the third case.

Summarizing we have

$$\frac{d}{dt}\int_0^1 \left(\frac{4}{5}r^5 - \frac{9}{4}r^4 + \frac{5}{2}r^2\right) u_r\, dr = \int_0^1 (6r-9)(u_r)^2\, r\, dr - 36\int_0^1 u_r\, r\, dr.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\int_0^1 \left(\frac{4}{5}r^5 - \frac{9}{4}r^4 + \frac{5}{2}r^2\right) u_r\, dr \quad &\leq \quad -\int_0^1 (9-6r)(u_r)^2\, r\, dr \\ &\qquad +36\frac{C^2}{2\epsilon} + 36\frac{\epsilon}{2}\int_0^1 (9-6r)(u_r)^2\, r\, dr \\ &\leq \quad -C\int_0^1 (9-6r)(u_r)^2\, r\, dr + C',\end{aligned}$$

where we have used

$$\begin{aligned}\int_0^1 u_r\, r\, dr \quad &= \quad \int_0^1 \frac{\sqrt{9-6r}}{\sqrt{9-6r}} u_r\, r\, dr \\ &\leq \quad \left(\int_0^1 \frac{1}{9-6r} r\, dr\right)^{1/2} \left(\int_0^1 (9-6r)(u_r)^2\, r\, dr\right)^{1/2} \\ &\leq \quad C\left(\int_0^1 (9-6r)(u_r)^2\, r\, dr\right)^{1/2} \\ &\leq \quad \frac{C^2}{2\epsilon} + \frac{\epsilon}{2}\left(\int_0^1 (9-6r)(u_r)^2\, r\, dr\right),\end{aligned}$$

and here we have employed Hölder inequality and Young inequality in the first and third inequalities respectively.

We finish our proof with the estimate

$$\begin{gathered}\frac{d}{dt}\int_0^1 \left(\frac{4}{5}r^4 - \frac{9}{4}r^3 + \frac{5}{2}r\right) u_r\, r\, dr \leq \\ -C\int_0^1 \left(\frac{4}{5}r^4 - \frac{9}{4}r^3 + \frac{5}{2}r\right)^2 (u_r)^2\, r\, dr + C' \leq \\ -C\left[\int_0^1 \left(\frac{4}{5}r^4 - \frac{9}{4}r^3 + \frac{5}{2}r\right) u_r\, r\, dr\right]^2 + C',\end{gathered}$$

where we have used that $9-6r$ is bounded from below by a positive constant and $4r^4/5 - 9r^3/4 + 5r/2$ is non-negative and bounded from above in $[0,1]$ in the first step and Jensen inequality in the second step. This automatically implies blow-up in finite time

$$\int_0^1 \left(\frac{4}{5}r^4 - \frac{9}{4}r^3 + \frac{5}{2}r\right) u_r\, r\, dr \to -\infty \quad \text{when} \quad t \to T^{**}, \tag{67}$$

for $T^{**} < \infty$ and a sufficiently large initial condition. In turn, this proves that the solution ceases to exist at some time $T^* \leq T^{**}$. □

Remark 5.3. A subtle distinction should be made between solutions which "cease to exist" and solutions which "blow up". The former concerns the existence of the smooth solution, the latter concerns the unboundedness of some norm; whence the latter implies the former. Theorem 5.2 merely states that the smooth solution ceases to exist, with no statement about blow-up. In particular, if (67) held then an integration by parts would show that the $L^1(B_1(0))$ norm of the solution would blow up. We conjecture that this is not true and that, in fact, $T^* < T^{**}$.

We conclude this paper with some natural questions and some open problems.

• **Uniqueness and/or multiplicity of stationary solutions.**
By [11] we know that (9) admits the trivial solution $u \equiv 0$ and also a mountain pass solution $\widetilde{u}$. One can then wonder whether (9) also admits further solutions. Note first that the functional J in (10) is not even and, therefore, standard multiplicity results are not available. In particular, $-\widetilde{u}$ is not a solution to (9). Does the multiplicity of solutions depend on the domain? Can we have finite multiplicity results? What about radial solutions in the ball? In this case, one can refer to some results in [8, 9]. An answer to these questions would lead to an improvement of the statement of Theorem 4.10, by making more precise the possible structure of the ω-limit set $\omega(u_0)$, see Remark 4.11.

• **Blow up in L^p norms.**
From Theorems 4.8 and 4.10 we learn that when blow up occurs, then also the $W_0^{1,4}(\Omega)$-norm blows up. What about the $L^p(\Omega)$-norms? Is there some critical exponent $q \in (1, \infty)$ such that the blow up in the $L^p(\Omega)$-norm occurs if and only if $p > q$? Or does the $L^\infty(\Omega)$-norm remain bounded? And, even more interesting, what happens under Navier boundary conditions? In this respect, see Theorem 5.2 and Remark 5.3.

• **Qualitative properties of solutions.**
It is well-known that the biharmonic operator under Dirichlet boundary conditions does not satisfy the positivity preserving property in general domains, see e.g. [16]. Moreover, also the biharmonic heat operator in $\mathbb{R}^n$ does not preserve positivity and exhibits only *eventual local positivity*, see [12, 15] where also nonlinear problems are considered. For there reasons, a full positivity preserving property for (36) (such as $u_0 \geq 0$ implies $u(t) \geq 0$) cannot be expected. However, one can wonder whether (36) has some weaker form of positivity preserving, for instance bounds for the negative part of $u(t)$ when $u_0 \geq 0$.

• **Other boundary conditions.**
According to the physical model one wishes to describe, it could be of interest to study (36) with different boundary conditions. In particular, it could be interesting to consider the more deeply the Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. For the stationary problem (9), these conditions were studied in [8, 9, 11]. It turns out that (9) is no longer of variational type and different techniques (such as fixed point theorems) need to be employed. Therefore, it is not clear whether an energy functional can be defined and if the same proofs of the present paper may be applied. More generally, one could also consider the so-called Steklov boundary

conditions $u = \Delta u - a u_\nu = 0$ on $\partial\Omega$, where $a \in C(\partial\Omega)$ should take into account the mean curvature of the (smooth) boundary. We refer to [16] for the derivation and the physical meaning of these conditions.

• Further regularity of the solution.

Using the regularizing effect of the biharmonic heat operator, one could wonder which (maximal) regularity should be expected for solutions to (36). Moreover, since the nonlinearity $u \mapsto \det(D^2u)$ is analytic, one could investigate if the results in [19, 33] concerning the ω-limit can be extended to (36). This would allow to improve Theorem 4.10 and to have convergence of the whole flow $u(t)$ and not only of a subsequence. However, this appears nontrivial due to the presence of different second order derivatives involved.

• High energy initial data.

Except for Theorem 4.7, in order to prove global existence or finite time blow-up for (36) we assumed that $J(u_0) \le d$. What happens for $J(u_0) > d$? Possible hints may be found in [14, 17] although the lack of a comparison principle for (9) certainly creates more difficulties. Can the basin of attraction of the trivial solution $u \equiv 0$ be characterized more explicitly?

• Higher space dimensions.

If we set the equation (36) in some $\Omega \subset \mathbb{R}^n$ with $n \ge 3$ we lose the physical application but the problem is mathematically challenging. If $n \le 4$ the embedding $W^{2,2}_0(\Omega) \subset W^{1,4}_0(\Omega)$ is still true, although for $n = 4$ it becomes a critical embedding which lacks compactness. Moreover, the embedding $W^{2,2}_0(\Omega) \subset L^\infty(\Omega)$ fails for $n \ge 4$. But the most relevant problem concerns the nonlinearity $\det(D^2u)$ which has the same degree as the dimension. For instance, if $n = 3$ the term $\det(D^2u)$ is cubic, involving the product of 3 second order derivatives. Since each derivative merely belongs to $L^2(\Omega)$ (whenever $u \in W^{2,2}_0(\Omega)$), this term may not belong to any L^p space. Hence, no variational approach can be used and a different notion of solution is needed.

REFERENCES

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, *Some remarks on elliptic problems with critical growth in the gradient*, J. Diff. Eq. **222** (2006) 21–62.

[2] B. Abdellaoui, A. Dall'Aglio, I. Peral, *Regularity and nonuniqueness results for parabolic problems arising in some physical models, having natural growth in the gradient*, J. Math. Pures Appl. **90** (2008) 242–269.

[3] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14**, (1973) 349–381.

[4] A.L. Barabási, H.E. Stanley, *Fractal Concepts in Surface Growth*, Cambridge University Press, Cambridge (1995).

[5] R. Coifman, P. L. Lions, Y. Meyer, S. Semmes, *Compensated compactness and Hardy spaces*, J.Math. Pures Appl. **72** (1993) 247–286.

[6] Der-Chen Chang, G. Dafni, E.M. Stein, *Hardy Spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in $\mathbb{R}^N$*, Trans. Amer. Math. Soc. **351** (1999) 1605–1661.

[7] C. Escudero, *Geometric principles of surface growth*, Phys. Rev. Lett. **101** (2008) 196102.

[8] C. Escudero, R. Hakl, I. Peral, P.J. Torres, *On radial stationary solutions to a model of nonequilibrium growth*, Eur. J. Appl. Math. **24** (2013) 437–453.

[9] C. Escudero, R. Hakl, I. Peral, P.J. Torres, *Existence and nonexistence results for a singular boundary value problem arising in the theory of epitaxial growth*, Mathematical Methods in the Applied Sciences (published online).

[10] C. Escudero, E. Korutcheva, *Origins of scaling relations in nonequilibrium growth*, J. Phys. A: Math. Theor. **45** (2012) 125005.

[11] C. Escudero, I. Peral, *Some fourth order nonlinear elliptic problems related to epitaxial growth*, J. Differential Equations **254** (2013) 2515–2531.

[12] A. Ferrero, F. Gazzola, H.C. Grunau, *Decay and eventual local positivity for biharmonic parabolic equations*, Disc. Cont. Dynam. Syst. **21** (2008) 1129–1157.

[13] E. Gagliardo, *Proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **7** (1958) 102–137.

[14] F. Gazzola, *Finite time blow-up and global solutions for some nonlinear parabolic equations*, Diff. Int. Eq. **17** (2004) 983–1012.

[15] F. Gazzola, H.C. Grunau, *Eventual local positivity for a biharmonic heat equation in* $\mathbb{R}^n$, Disc. Cont. Dynam. Syst. S **1** (2008) 83–87.

[16] F. Gazzola, H.C. Grunau, G. Sweers, *Polyharmonic boundary value problems*, LNM 1991 Springer (2010).

[17] F. Gazzola, T. Weth, *Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level*, Diff. Int. Eq. **18** (2005) 961–990.

[18] M. Grasselli, G. Mola, A. Yagi, *On the longtime behavior of solutions to a model for epitaxial growth*, Osaka J. Math. **48** (2011) 987–1004.

[19] M.A. Jendoubi, *A simple unified approach to some convergence theorems of L. Simon*, J. Funct. Anal. **153** (1998) 187–202.

[20] M. Kardar, G. Parisi, Y.C. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986) 889–892.

[21] R.V. Kohn, T.S. Lo, *A new approach to the continuum modeling of epitaxial growth: slope selection, coarsening, and the role of the uphill current*, Physica D **161** (2002) 237–257.

[22] R.V. Kohn, X. Yan, *Upper bound on the coarsening rate for an epitaxial growth model*, Comm. Pure Appl. Math. **LVI** (2003) 1549–1564.

[23] Z.W. Lai, S. Das Sarma, *Kinetic growth with surface relaxation: Continuum versus atomistic models*, Phys. Rev. Lett. **66** (1991) 2348–2351.

[24] H.A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$, Arch. Rational Mech. Anal. **51** (1973) 371–386.

[25] B. Li, J.G. Liu, *Epitaxial growth without slope selection: energetics, coarsening, and dynamic scaling*, J. Nonlinear Sci. **14** (2004) 429–451.

[26] M. Marsili, A. Maritan, F. Toigo, J. R. Banavar, *Stochastic growth equations and reparametrization invariance*, Rev. Mod. Phys. **68** (1996) 963–983.

[27] S. Müller, *Det=det. A remark on the distributional determinant*, C.R. Acad. Sci. Paris, Série I **311** (1990) 13–17.

[28] W.M. Ni, P.E. Sacks, J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Diff. Eq. **54** (1984) 97–120.

[29] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa **13** (1959) 115–162.

[30] L.E. Payne, D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975) 273–303.

[31] P. Poláčik, F. Simondon, *Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains*, J. Diff. Eq. **186** (2002) 586–610.

[32] P. Quittner, P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states.* Birkhäuser Advanced Texts, Basler Lehrbücher. Basel (2007).

[33] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. Math. **118** (1983) 525–571.

[34] T. Sun, H. Guo, M. Grant, *Dynamics of driven interfaces with a conservation law*, Phys. Rev. A **40** (1989) R6763–R6766.

[35] M. Tsutsumi, *On solutions of semilinear differential equations in a Hilbert space*, Math. Japon. **17** (1972) 173–193.
[36] J. Villain, *Continuum models of crystal growth from atomic beams with and without desorption*, J. Phys. I (France) **1** (1991) 19–42.

Carlos Escudero
Departamento de Matemáticas & ICMAT (CSIC-UAM-UC3M-UCM)
Universidad Autónoma de Madrid
carlos.escudero@uam.es

Filippo Gazzola
Dipartimento di Matematica
Politecnico di Milano
filippo.gazzola@polimi.it

Ireneo Peral
Departamento de Matemáticas
Universidad Autónoma de Madrid
ireneo.peral@uam.es