On a nonlinear nonlocal hyperbolic system modeling suspension bridges

Gianni ARIOLI [♯] - Filippo GAZZOLA [†]

[♯] MOX– Modellistica e Calcolo Scientifico
[♯,†] Dipartimento di Matematica – Politecnico di Milano
Piazza Leonardo da Vinci 32 - 20133 Milano, Italy
gianni.arioli@polimi.it - filippo.gazzola@polimi.it

Abstract

We suggest a new model for the dynamics of a suspension bridge through a system of nonlinear nonlocal hyperbolic differential equations. The equations are of second and fourth order in space and describe the behavior of the main components of the bridge: the deck, the sustaining cables and the connecting hangers. We compute all the energies involved and we derive the equations from variational principles. We then prove existence and uniqueness of a weak solution for the resulting problem.

1 Introduction

The main span of a suspension bridge is a complex structure composed by several interacting components, see Figure 1. Four towers sustain two cables that, in turn, sustain the hangers. At their lower endpoint the hangers are linked to the deck and sustain it from above. The deck is represented by a thin rectangular plate. The hangers are hooked to the cables and the deck is hooked to the hangers; the weight of the deck deforms the cable and stretches the hangers that exert a restoring action on the deck.

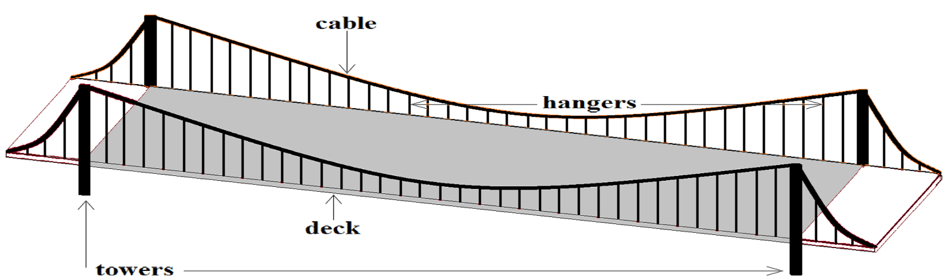

Figure 1: Sketch of a suspension bridge.

This complex structure is extremely interesting from a mathematical point of view. It appears quite challenging to find a model simple enough to be mathematically tractable and sufficiently accurate to display the main features of a real bridge. The celebrated report by Navier [23] has been for about one century the only mathematical treatise of suspension bridges. Much later, the deflection theory for beams was introduced in the monograph by Melan [21]. A further historical source is the work by

Smith-Vincent [26] where the bridge is again modeled as a one dimensional beam: after linearization they obtain a fourth order linear ODE [26, (4.2)] which can be integrated explicitly.

The two major problems of suspension bridges are their manifest nonlinear behavior [9] and their instability, in particular they are prone to torsional oscillations. This is apparent in videos available on the web [27] as well as in many other events, see e.g. [11] for a survey. Therefore, a reliable model for a suspension bridge should be able to display torsional oscillations and a beam-type equation does not achieve this goal.

A first model able to display torsional oscillations was introduced by McKenna [17, 19] who derives a system of coupled ODE's; this model views the cross section of the deck as a rod subject to the forces exerted by two lateral nonlinear springs representing the hangers. A model involving vertical oscillations of the deck, its torsional angle, and coupling with the two sustaining cables was suggested by Matas-Očenášek [16] who consider the hangers as linear springs and obtain a system of four PDE's, see also (SB_4) in [10].

Partial explanations of the instability in bridges are based on aerodynamic effects such as vortex shedding, flutter, parametric resonance, aerodynamic forces, see [6]. But none of these theories explains how a longitudinal oscillation can rapidly switch into a torsional one. Scanlan [25, p.209] writes that the switch is due to *some fortuitous condition*, a justification which does not seem very scientific. And, indeed, McKenna [18] raises many doubts about the mathematical models and motivations used in classical literature.

In a recent paper [2] we suggested that the onset of the torsional instability could be of purely structural nature. The model adopted there was oversimplified and did not describe accurately the nonlinear behavior of the cables+hangers restoring force. By using Poincaré-type maps we were able to show that a longitudinal oscillation in an ideally isolated bridge may switch rapidly into a torsional oscillation when enough energy is present within the structure.

Aiming to display the phenomenon of transfer of energy between longitudinal and torsional oscillations as in an actual bridge, in this paper we suggest a new and realistic model which takes into account all the main components of the bridge and its quantitative structural parameters. We compute the energies and we derive the corresponding Euler-Lagrange equations by variational principles. The dynamics of the bridge is modeled through a system of four hyperbolic equations, three of them of second order in space and one of fourth order. This system also contains nonlinear couplings (the restoring action of the hangers) and nonlocal terms (the elongation of the extensible cables). In fact, the full energy balance is rather complicated, therefore we approximate the equations with a first order expansion and we prove that the initial-boundary value problem for the hyperbolic system admits a unique solution. In a forthcoming paper [3] we will study higher order approximations.

In Section 2 we derive an accurate model and we proceed in two steps. First, in Sections 2.1 and 2.2 we compute the Lagrangian of a system composed by a one-dimensional beam suspended to a cable through hangers, see Figure 2. In the classical literature [7, 15, 21] one neglects either the mass of the cable, in which case the load is distributed per horizontal unit and the cable takes the shape of a parabola, or the mass of the beam, in which case the load is distributed per unit length and the cable takes the shape of a catenary. Since none of these masses is negligible, we take here into account both their contributions. This gives rise to a somehow intermediate shape and the linearized problem becomes a Sturm-Liouville problem with a weight. Second, in Section 2.3 we extend the result to the full suspension bridge by modeling the deck as a degenerate plate, that is, a beam representing the midline of the deck with cross sections which are free to rotate around the beam; simplified equations representing this model were previously analyzed in [5, 22]. The midline corresponds to the barycenters of the cross sections. The cross sections are seen as rods that can rotate around their barycenter. The endpoints of the cross sections are the edges of the plate that are connected to the sustaining cables through the hangers.

Then we derive the Euler-Lagrange equations by variational methods, that is, by finding critical points of the Lagrangian. The resulting equations form a system of semilinear hyperbolic equations, three of them being of second order and one of fourth order, see (38). All the coefficients are continuous time-independent functions and (38) is a nonlocal differential problem due to the presence of the integral term which takes into account the cables elongations. A nonlinear term is responsible for the coupling between the equations. These peculiarities show that (38) is not a standard problem and therefore we need to prove existence and uniqueness of a solution. This is done in Section 3.

2 The physical model

Throughout this paper we denote the partial derivatives of a function $y = y(x,t)$ by

$$y' = \frac{\partial y}{\partial x}\,, \qquad \dot{y} = \frac{\partial y}{\partial t}$$

and similarly for higher order derivatives.

2.1 A beam sustained by a cable through hangers

As a first step, we derive the Lagrangian of a suspended beam connected to a sustaining cable through hangers. Let w be the position of the beam and $p - s$ be the position of the cable, as in Figure 2. The purpose of this section is to derive the equation of this cable-hangers-beam system.

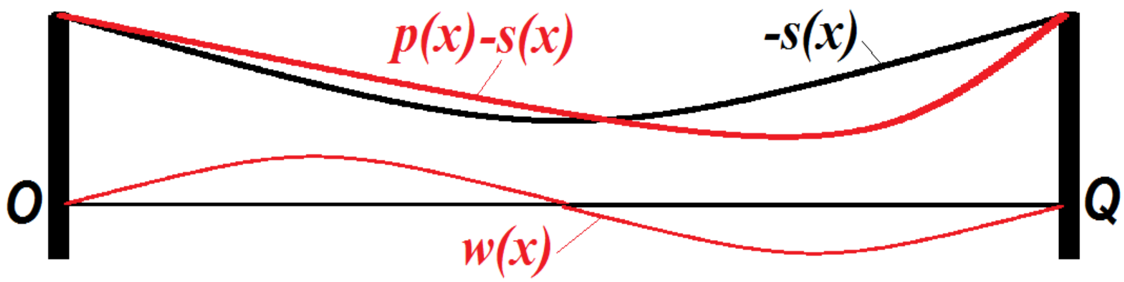

Figure 2: Cable-beam structure modeling a suspension bridge.

We model the cable as a perfectly flexible string subject to vertical loads. If we assume that the string has no resistance to bending, the only internal force is the tension $F = F(x)$ of the string which acts tangentially to the position of the curve representing the string. In Figure 3 we sketch a picture of

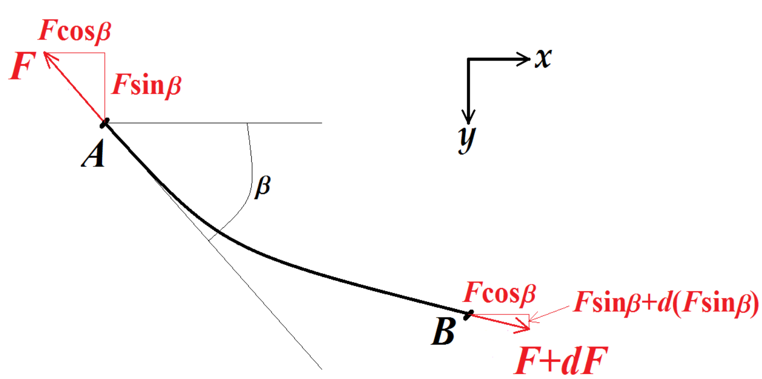

Figure 3: Equilibrium of a string.

the string whose endpoints are A and B and whose position is described by a function $-s(x) < 0$, the origin being below B. The horizontal direction represents the abscissa x whereas the downwards axis represents the vertical displacement. Denote by $\beta = \beta(x)$ the angle between the horizontal x-direction and the tangent to the curve so that

$$-s'(x) = \tan \beta(x) . \tag{1}$$

The horizontal component of the tension H_0 is constant, that is,

$$F(x) \cos \beta(x) \equiv H_0 > 0 . \tag{2}$$

The classical deflection theory of suspension bridges models the bridge structure as a combination of a string (the sustaining cable) and a beam (the deck). The cable carries its own weight, the weight of the hangers (which we assume to be negligible) and the weight of the beam. Possible deformations of the cable and the beam are assumed to be small, that is, we aim to model small displacements of the beam and the cable with respect to the equilibrium position; therefore we assume that

$$\text{all the displacements and their derivatives are small.} \tag{3}$$

In Figure 2 we represent the position $-s(x)$ of the cable at rest with $-s_0 < 0$ being the level of the left and right endpoints of the cable (s_0 is the height of the towers). We denote by L the distance between the towers. Assume that a beam of length L and linear density of mass M is hanged to the cable whose linear density of mass is m: the segment OQ represents the horizontal position of the beam at rest. Since the spacing between hangers is small relative to the span, the hangers can be considered as a continuous membrane connecting the cable and the beam. Then the cable is subject to a downwards vertical force given by

$$q(x) = \left(M + m\sqrt{1 + s'(x)^2} \right) g$$

g being the gravitational constant. In this situation, the vertical component of the tension has a variation given by

$$\frac{d}{dx}[F(x) \sin \beta(x)] = -q(x)$$

where we recall that the positive vertical axis is oriented downwards. In view of (2) we then obtain

$$H_0 \frac{d}{dx}[\tan \beta(x)] = -q(x)$$

Moreover, using (1) and taking into account that the cable is hanged to two towers of same height s_0, we get

$$\begin{cases} H_0 s''(x) = \left(M + m\sqrt{1 + s'(x)^2} \right) g , \\ s(0) = s(L) = s_0 \end{cases} \tag{4}$$

If one neglects the mass of the cable ($m = 0$) then the solution is the parabola

$$s_p(x) = s_0 - \frac{Mg}{2H_0} x(L - x) , \tag{5}$$

while if one neglects the mass of the beam ($M = 0$) the solution is the catenary

$$s_c(x) = \frac{H_0}{mg} \left[\cosh \left(\frac{mg}{2H_0}(2x - L) \right) - \cosh \left(\frac{mgL}{2H_0} \right) \right] + s_0 .$$

We point out that if we take $m = 0$ in the above explicit equations and we assume that the sag-span ratio for is $1/12$ (a typical value for actual bridges, see e.g. [24, §15.17]), by using (5) we have

$$\frac{gM}{2H_0} = \frac{2}{3L}$$

and then

$$0 < s_p(x) - s_c(x) \leq s_p(L/2) - s_c(L/2) \approx 6 \cdot 10^{-3}\, L \qquad \forall x \in (0, L)\,.$$

Hence, for a typical length of the bridge $L = 1km$ the maximum difference between the two configurations is about $6m$.

Our purpose is to consider both the masses of the beam and the cable ($m, M > 0$). Then (4) does not have simple explicit solutions. Moreover, this second order equation has no variational structure and the problem has boundary conditions, therefore existence and uniqueness of the solution are not granted, but are proved in the next statement.

Proposition 1. *For any $m, M > 0$ there exists a unique solution $s = s(x)$ of (4); this solution is symmetric with respect to $x = L/2$, that is, $s(x) = s(L - x)$.*

Proof. We first prove the existence and uniqueness of a symmetric solution of (4), then we show that any solution of (4) is symmetric. These two steps will prove the statement.

For any $s_1 \in \mathbb{R}$ the initial value problem

$$\begin{cases} H_0 s''(x) = \left(M + m\sqrt{1 + s'(x)^2}\right) g\,, \\ s(L/2) = s_1\,, \\ s'(L/2) = 0\,, \end{cases} \tag{6}$$

has a unique solution defined on the whole real line for all values of s_1. Moreover, solutions of (6) with different values of s_1 differ by a constant and they are all symmetric with respect to $x = L/2$. Then there exists a unique $s_1 < s_0$ such that the solution of (6) satisfies $s(0) = s(L) = s_0$: this solution is the unique symmetric solution of (4).

Take now a solution $\overline{s}$ of (4) which we know to exist in view of what we just proved. Since $\overline{s}$ is convex and $\overline{s}(0) = \overline{s}(L)$, by the Lagrange Theorem there exists a unique $x_0 \in (0, L)$ such that $\overline{s}'(x_0) = 0$. Hence, $\overline{s}$ solves the initial value problem

$$\begin{cases} H_0 s''(x) = \left(M + m\sqrt{1 + s'(x)^2}\right) g\,, \\ s(x_0) = \overline{s}(x_0)\,, \\ s'(x_0) = 0\,. \end{cases} \tag{7}$$

But also $\overline{s}(2x_0 - x)$ solves (7): then by uniqueness of the solution we infer that $\overline{s}(x) \equiv \overline{s}(2x_0 - x)$. Since $\overline{s}(0) = \overline{s}(L) = s_0$ and since $\overline{s}$ is convex, we finally deduce that $2x_0 = L$. Therefore, any solution $\overline{s}$ of (4) satisfies $\overline{s}(x) = \overline{s}(L - x)$. □

Hence, for general $m, M > 0$ the cable takes the shape of a curve with a ∪-shaped graph, something in between a parabola and a catenary. The length at rest of the cable is

$$L_c = \int_0^L \sqrt{1 + s'(x)^2}\, dx\,. \tag{8}$$

Let

$$\xi(x) := \sqrt{1 + s'(x)^2} \tag{9}$$

be the local length of the cable, which is related to the unique solution of (4) (see Proposition 1) and will often appear in the computations. In classical literature this term is usually approximated with $\xi(x) \equiv 1$. While modeling suspension bridges, Biot and von Kármán [7, p.277] warn the reader by writing *...whereas the deflection of the beam may be considered small, the deflection of the string, i.e., the deviation of its shape from a straight line, has to be considered as of finite magnitude.* Whence, (3) is acceptable whereas no approximation of $s'(x)$ should be taken. Nevertheless, at a subsequent stage, Biot-von Kármán [7, (5.14)] decide to *neglect* $s'(x)^2$ *in comparison with unity*: this leads to $\xi(x) \equiv 1$. As shown in [13], this choice may significantly change the responses of the system. Needless to say, this rough approximation considerably simplifies all the computations.

2.2 Lagrangian of the cable-hangers-beam system

In this section we write all the components of the Lagrangian of the single cable-hangers-beam system. Since the energies involved are complicated, we will use (3) in order to approximate them with asymptotic expansions up to second order terms. This will give rise to Euler-Lagrange equations with linear terms.

• **Kinetic energy of the beam.** If $w = w(x,t)$ denotes the position of the beam, then the kinetic energy of the beam is given by

$$E_{kb} = \frac{M}{2}\int_0^L \dot{w}^2\, dx\,. \tag{10}$$

• **Kinetic energy of the cable.** If $p(x,t) - s(x)$ denotes the position of the cable, then the kinetic energy of the cable is given by

$$E_{kc} = \frac{m}{2}\int_0^L \dot{p}^2 \xi(x)\, dx\,. \tag{11}$$

• **Gravitational energy.** If we neglect the small extensions of the cable, the gravitational energy is given by

$$E_g = -g\int_0^L \Big(Mw + mp\xi(x)\Big)\, dx\,.$$

The minus sign is due to the downward orientation of the vertical axis.

• **Elastic energy in the hangers.** The hangers behave as stiff linear springs when in tension and they do not apply any force when compressed. The latter case is called *slackening of the hangers*. Let $\lambda = \lambda(x)$ be the length of the unloaded hanger at position $x \in (0,L)$; this is the vertical length of the hanger when it is fixed to the sustaining cable and *before* hanging the beam. After the beam is installed, the hangers are in tension and reach a new length $s(x) > \lambda(x)$, where s is the solution of (4); if no additional load acts on the system, the equilibrium position of the beam is $w(x) \equiv 0$ (corresponding to the segment OQ in Figure 2) and the position of the cable at equilibrium (beam installed) is $-s(x)$. As one expects from a linear spring, the Hooke constant $\kappa(x)$ of each hanger is proportional to the inverse of its unloaded length, so that

$$\kappa(x) = \frac{\kappa_0}{\lambda(x)} \qquad (\kappa_0 > 0)\,.$$

Then the (linear density of) force due to the hanger in position x is

$$\kappa(x)\Big(s(x) - \lambda(x)\Big)\,. \tag{12}$$

Since the (linear density of) weight of the beam is Mg, we find

$$Mg = \kappa(x)\Big(s(x) - \lambda(x)\Big) = \frac{\kappa_0\Big(s(x) - \lambda(x)\Big)}{\lambda(x)} \tag{13}$$

which shows that the relative elongation of the hangers is proportional to the density of weight after the beam is installed: this relationship readily gives the following dependence of the elongation $s(x)$ on the unloaded length

$$s(x) = \left(1 + \frac{Mg}{\kappa_0}\right)\lambda(x)\,.$$

The actual length of the hanger is $w(x,t) - (p(x,t) - s(x))$, so that the hanger applies a force

$$\begin{aligned} F(w-p) &= \kappa(x)\Big(w(x,t) - (p(x,t) - s(x)) - \lambda(x)\Big)^+ = \kappa(x)\Big(w(x,t) - p(x,t) + \frac{Mg}{\kappa(x)}\Big)^+ \\ &= \Big((\kappa_0 + Mg)\frac{w(x,t) - p(x,t)}{s(x)} + Mg\Big)^+ ; \end{aligned} \tag{14}$$

note that $F(0) = Mg$, that is, the restoring force of the hangers balances gravity at equilibrium. The positive part in (14) models the fact that the hangers behave as linear springs when extended and do not yield any force when slackened. The equalities in (14) are due to (13). The elastic energy is then given by

$$\begin{aligned} E_h(w-p) &= \frac{1}{2}\int_0^L \kappa(x)\bigg(\Big(w(x,t) - p(x,t) + \frac{Mg}{\kappa(x)}\Big)^+\bigg)^2 dx \\ &= \frac{1}{2}\int_0^L \kappa(x)\bigg(\Big(w(x,t) - p(x,t) + s(x) - \lambda(x)\Big)^+\bigg)^2 dx\,. \end{aligned} \tag{15}$$

Remark 1. The special form of F in (14) was first suggested by McKenna-Walter [20] and well models the possible slackening of the hangers, at least in a first order approximation as in the present paper. Brownjohn [9] claims that the slackening mechanism is not as simple as an on/off force, as described by the positive part of the elongation. Our results remain true if the force F is replaced by any Lipschitz continuous function. □

• **Bending energy of the beam.** The bending energy of a beam depends on its curvature, see [7] and also [12] for a more recent approach and further references. The Young modulus E is a measure of the stiffness of an elastic material and is defined as the ratio stress/strain. If Idx denotes the moment of inertia of a cross section of length dx, then the constant quantity EI is the flexural rigidity of the beam. The energy necessary to bend the beam is the square of the curvature times half the flexural rigidity:

$$E_B = \frac{EI}{2}\int_0^L \frac{(w'')^2}{(1+(w')^2)^3}\sqrt{1+(w')^2}\,dx \tag{16}$$

where we highlighted the curvature and the arclength. This energy is not convex and it fails to be coercive in any reasonable functional space so that standard methods of calculus of variations do not apply. Assuming (3), an asymptotic expansion in (16) yields

$$E_B = \frac{EI}{2}\int_0^L \frac{(w'')^2}{(1+(w')^2)^{5/2}}\,dx \approx \frac{EI}{2}\int_0^L (w'')^2\left(1 - \frac{5}{2}(w')^2\right)dx = \frac{EI}{2}\int_0^L (w'')^2\,dx + o\Big((w'')^2\Big)\,; \tag{17}$$

note that the neglected infinitesimal term in (17) is also $o\Big((w')^2\Big)$ in view of (3). Therefore, from now on, we simply take

$$E_B = \frac{EI}{2}\int_0^L (w'')^2\,dx\,.$$

• **Stretching energy of the beam.** If large deformations are involved, the strain-displacement relation is not linear and a possible nonlinear model was suggested by Woinowsky-Krieger [31]: he modified the classical Bernoulli-Euler beam theory by assuming a nonlinear dependence of the axial strain on the deformation gradient and by taking into account the stretching of the beam due to its elongation in the longitudinal direction. In this situation there is a coupling between bending and stretching and the stretching energy is proportional to the elongation of the beam which results in

$$E_S = \frac{\gamma}{2}\left(\int_0^L \left(\sqrt{1+(y')^2}-1\right)dx\right)^2 \approx \frac{\gamma}{8}\left(\int_0^L (y')^2\,dx\right)^2$$

where $\gamma > 0$ is the elastic constant of the beam. Since this term is of fourth order, in the sequel we drop it.

• **Stretching energy of the cable.** The tension of the cable consists of two parts, the tension at rest

$$H(x) = H_0\xi(x) \tag{18}$$

and the additional tension

$$\frac{AE}{L_c}\,\Gamma(p)$$

due to the increment of length $\Gamma(p)$ of the cable. The latter requires the energy

$$E_{tc}(p) = \frac{AE}{2L_c}\,\Gamma(p)^2\,. \tag{19}$$

Assuming (3), we may approximate $E_{tc}(p)$ as follows. First, we note the asymptotic expansion

$$\sqrt{1+[s'(x)-p'(x)]^2}-\sqrt{1+s'(x)^2} = -\frac{s'(x)p'(x)}{\xi(x)} + \frac{p'(x)^2}{2\xi(x)^3} + o\Big(p'(x)^2\Big)\,. \tag{20}$$

Then we approximate the increment $\Gamma(p)$ of the length of the cable, due to its displacement p, by using (20):

$$\Gamma(p) := \int_0^L \sqrt{1+[s'(x)-p'(x)]^2}\,dx - L_c \approx \int_0^L \left(\frac{p'(x)^2}{2\xi(x)^3} - \frac{s'(x)p'(x)}{\xi(x)}\right)dx\,.$$

By squaring we find

$$\Gamma(p)^2 = \left(\int_0^L \frac{s'(x)p'(x)}{\xi(x)}\,dx\right)^2 + o\Big((p')^2\Big)\,.$$

Therefore, we approximate (19) by

$$E_{tc}(p) = \frac{AE}{2L_c}\left(\int_0^L \frac{s'(x)p'(x)}{\xi(x)}\,dx\right)^2\,.$$

By taking the variation of this energy and integrating by parts we obtain

$$dE_{tc}(p)z = -\frac{AE}{L_c}\left(\int_0^L \frac{s'(x)p'(x)}{\xi(x)}\,dx\right)\int_0^L \frac{s''(x)z(x)}{\xi(x)^3}\,dx + o(p)\,. \tag{21}$$

On the other hand, the amount of energy needed to deform the cable at rest under the tension (18) in the infinitesimal interval $[x, x+dx]$ from the original position $-s(x)$ to $-s(x)+p(x)$ is the variation of length times the tension, that is

$$E(x)\,dx = H_0\xi(x)\left(\sqrt{1+[s'(x)-p'(x)]^2}-\sqrt{1+s'(x)^2}\right)dx\,,$$

hence, using (20) and neglecting the higher order terms, the energy necessary to deform the whole cable is

$$E(p) = \int_0^L E(x)\,dx = H_0 \int_0^L \left(-s'(x)p'(x) + \frac{p'(x)^2}{2\xi(x)^2} \right) dx\,. \tag{22}$$

Then, the variation of energy $\sigma^{-1}[E(p+\sigma z) - E(p)]$ as $\sigma \to 0$ and some integration by parts lead to

$$dE(p)z = H_0 \int_0^L \left(-\, s'(x)z'(x) + \frac{p'(x)z'(x)}{\xi(x)^2} \right) dx = H_0 \int_0^L \left(s'' - \frac{p''}{\xi(x)^2} + \frac{2s's''p'}{\xi(x)^4} \right) z(x)\,dx\,. \tag{23}$$

• **The Euler-Lagrange equations.** The Lagrangian of the system is obtained as the sum of the kinetic energies minus the sum of the potential energies. To compute the Euler-Lagrange equations we derive the term

$$H_0 \left(s''(x) - \frac{p''(x)}{\xi(x)^2} + \frac{2s'(x)s''(x)p'(x)}{\xi(x)^4} \right) = \Big(M + m\xi(x) \Big)\, g - \frac{H_0}{\xi(x)^2} p''(x) + \frac{2H_0\, s'(x)s''(x)}{\xi(x)^4} p'(x) \tag{24}$$

from (23) and using (4). Then, from (21) we derive the additional tension in the cable $h(p)$ produced by its displacement p:

$$h(p) = -\frac{AE}{L_c} \left[\int_0^L \frac{s'(x)p'(x)}{\xi(x)} dx \right] \frac{s''(x)}{\xi(x)^3}\,. \tag{25}$$

Remark 2. As already mentioned, in classical engineering literature [7, 21] one usually replaces $\xi(x) = \sqrt{1 + s'(x)^2}$ with 1 and neglects the mass of the cable ($m = 0$) so that one finds (5), that is, $s''(x) = \frac{Mg}{H_0}$; with these rough approximations, one can easily obtain also the *second order expansion* of $h(p)$, that is,

$$\begin{aligned} & \frac{AE}{2L_c} \left(\int_0^L p'(x)^2\,dx \right) s''(x) + \frac{AE}{L_c} \left(\int_0^L s'(x)p'(x)\,dx \right) (p''(x) - s''(x) - 3s'(x)s''(x)p'(x)) \\ = \; & \frac{AE}{2L_c} \left(\int_0^L p'(x)^2\,dx \right) s''(x) - \frac{AE}{L_c}\frac{Mg}{H_0} \left(\int_0^L p(x)\,dx \right) (p''(x) - s''(x) - 3s'(x)s''(x)p'(x)) \end{aligned}$$

where we only see the terms of the expansion up to order 2. However, in literature one finds several further simplifications. Timoshenko [29] (see also [30, Chapter 11]) manipulates the functional $\Gamma(p)$ and reaches the expression

$$\frac{AE}{L_c} \left(\int_0^L \left(\frac{Mg}{H_0} p(x) + \frac{p'(x)^2}{2} \right) dx \right) (s''(x) - p''(x))$$

see [30, (11.16)]. This expression fails to consider some second order terms but inserts some third order terms. Further simplifications can be found in literature; Biot-von Kármán [7, (5.14)] neglect the second term and simply obtain

$$\frac{AE}{L_c}\frac{Mg}{H_0} \left(\int_0^L p(x)\,dx \right) (s''(x) - p''(x))\,.$$

This clearly simplifies many computations but this approximation seems not to have a sound justification. This is one further reason to stick to a *first order approximation* of $h(p)$. □

Let F be the force in (14), then by taking into account all the energy variations previously found we obtain the following equations of motion:

$$m\xi(x)\ddot{p} = \frac{H_0}{\xi(x)^2} p''(x) - \frac{2H_0\, s'(x)s''(x)}{\xi(x)^4} p'(x) - h(p) + F(w-p) - Mg\,, \tag{26}$$

$$M\ddot{w} = -EIw'''' - F(w-p) + Mg\,. \tag{27}$$

We emphasize that the term $mg\xi(x)$ has disappeared in (26) due to its appearance with the opposite sign in (24) and in the gravitational energy.

2.3 A model for the suspension bridge

In this subsection we consider the full suspension bridge. We view the deck as a degenerate plate, namely a beam representing the midline of the deck with cross sections which are free to rotate around the beam. The midline corresponds to the barycenters of the cross sections, which are seen as rods that can rotate around their barycenter, the angle of rotation with respect to the horizontal position is denoted by θ. The endpoints of the cross sections are the edges of the plate and they are connected to the sustaining cables through the hangers. Then, for small θ, the positions of the two free edges of the deck are given by

$$w = y \pm \ell \sin\theta \approx y \pm \ell\theta \,.$$

Each free edge of the deck is connected to a sustaining cable through hangers. We derive here the equation of this cable-hangers-deck system and, in order to derive the Euler-Lagrange equations, we start from the cable-hangers-beam model developed in the previous subsections.

We denote by $p_1(x,t)$ and $p_2(x,t)$ the displacements of the cables from their equilibrium position $s(x)$, see (4): hence, $p_i(x,t) - s(x)$ denotes the actual position of the i-th cable ($i = 1, 2$). We assume that the deck has length L and width 2ℓ with $2\ell \ll L$.

• **Kinetic energy of the deck.** This energy is composed by two terms, the first corresponding to the kinetic energy of the barycenter of the cross section, see (10), the second corresponding to the kinetic energy of the torsional angle. The kinetic energy of a rotating object is $\frac{1}{2}J\dot\theta^2$, where J is the moment of inertia and $\dot\theta$ is the angular velocity. The moment of inertia of a rod of length 2ℓ about the perpendicular axis through its center is given by $\frac{1}{3}M\ell^2 dx$ where Mdx is the mass of the rod. Hence, the kinetic energy of a rod having mass Mdx and half-length ℓ, rotating about its center with angular velocity $\dot\theta$, is given by $\frac{Mdx}{6}\ell^2\dot\theta^2$. Therefore, the kinetic energy of the deck is given by

$$E_{kd} = \int_0^L \left(\frac{M}{6}\ell^2\dot\theta^2 + \frac{M}{2}\dot y^2 \right) dx \,. \tag{28}$$

Note that now each cable sustains the weight of half deck, so the boundary value problem (4) for $s(x)$ becomes instead

$$H_0 s''(x) = \left(\frac{M}{2} + m\xi(x) \right) g \,, \qquad s(0) = s(L) = s_0 \,.$$

• **Kinetic energy of the cables.** We adopt (11) for both cables and we get

$$E_{kc} = \frac{m}{2}\int_0^L (\dot p_1^2 + \dot p_2^2)\, \xi(x)\, dx \,. \tag{29}$$

• **Stiffening energy of the deck.** It is composed by two terms, the bending energy of the central beam and the torsional energy. As in (17), we have

$$E_B = \frac{EI}{2}\int_0^L (y'')^2\, dx \,, \tag{30}$$

whereas the torsional stiffness of the deck is computed in terms of the derivative of the torsional angle:

$$E_T = \frac{GK}{2}\int_0^L (\theta')^2\, dx \,.$$

• **Gravitational energy of the deck and cables.** These are readily computed and read

$$-Mg\int_0^L y\, dx \qquad \text{and} \qquad -\, mg\int_0^L (p_1 + p_2)\xi(x)\, dx \,.$$

• **Elastic energy in the hangers.** Following (15), the energies of the two rows of hangers are

$$E_{h_1}(y+\ell\theta-p_1)=\frac{1}{2}\int_0^L\kappa(x)\Big(\Big(y(x,t)+\ell\theta(x,t)-p_1(x,t)+s(x)-\lambda(x)\Big)^+\Big)^2\,dx\,,$$

$$E_{h_2}(y-\ell\theta-p_2)=\frac{1}{2}\int_0^L\kappa(x)\Big(\Big(y(x,t)-\ell\theta(x,t)-p_2(x,t)+s(x)-\lambda(x)\Big)^+\Big)^2\,dx\,.$$

Since the weight Mg of the deck is now supported by two cables, each one supporting $Mg/2$, the force F in (14) acts now on the two edges of the deck and this gives rise to the two forces:

$$\begin{aligned}F(y+\ell\theta-p_1)&=\kappa(x)\Big(y(x,t)+\ell\theta(x,t)-p_1(x,t)+s(x)-\lambda(x)\Big)^+\\&=\kappa(x)\Big(y(x,t)+\ell\theta(x,t)-p_1(x,t)+\frac{Mg}{2\kappa(x)}\Big)^+\,,\\F(y-\ell\theta-p_2)&=\kappa(x)\Big(y(x,t)-\ell\theta(x,t)-p_2(x,t)+s(x)-\lambda(x)\Big)^+\\&=\kappa(x)\Big(y(x,t)-\ell\theta(x,t)-p_2(x,t)+\frac{Mg}{2\kappa(x)}\Big)^+\,,\end{aligned}$$

with $F(0)=Mg/2$. Moreover, instead of (13) we have

$$\frac{Mg}{2}=\kappa(x)\Big(s(x)-\lambda(x)\Big)=\frac{\kappa_0\Big(s(x)-\lambda(x)\Big)}{\lambda(x)}\,.$$

• **The Euler-Lagrange equations.** Let $h(p)$ be as in (25), then the equations of motion are

$$m\xi(x)\,\ddot p_1=\frac{H_0}{\xi(x)^2}p_1''-\frac{2H_0\,s'(x)s''(x)}{\xi(x)^4}p_1'-h(p_1)+F(y+\ell\theta-p_1)-\frac{Mg}{2}\tag{31}$$

$$m\xi(x)\,\ddot p_2=\frac{H_0}{\xi(x)^2}p_2''-\frac{2H_0\,s'(x)s''(x)}{\xi(x)^4}p_2'-h(p_2)+F(y-\ell\theta-p_2)-\frac{Mg}{2}\tag{32}$$

$$M\ddot y=-EIy''''-F(y+\ell\theta-p_1)-F(y-\ell\theta-p_2)+Mg\tag{33}$$

$$\frac{M}{3}\ell^2\ddot\theta=GK\theta''+\ell\,F(y-\ell\theta-p_2)-\ell\,F(y+\ell\theta-p_1).\tag{34}$$

• **Boundary conditions.** The degenerate plate is assumed to be hinged between the two towers while it is clear that the cross sections between the towers are fixed and cannot rotate. This results in the boundary conditions

$$y(0,t)=y(L,t)=y''(0,t)=y''(L,t)=\theta(0,t)=\theta(L,t)=0\quad\forall t\ge 0\,.\tag{35}$$

Since p_i-s denotes the actual position of the cables, we also have

$$p_i(0,t)=p_i(L,t)=0\quad\forall t\ge 0\,,\qquad(i=1,2).\tag{36}$$

3 Existence and uniqueness for the suspension bridge system

3.1 Definition of solution

Up to scaling, we may assume that $L=\pi$: this will simplify the spectral decomposition of the differential operator. We endow the Hilbert spaces $L^2(0,\pi)$, $H^1_0(0,\pi)$, $H^2\cap H^1_0(0,\pi)$ with, respectively, the scalar products

$$(u,v)_2=\int_0^\pi uv\ ,\quad (u,v)_{H^1}=\int_0^\pi u'v'\ ,\quad (u,v)_{H^2}=\int_0^\pi u''v''\ .$$

Then an orthogonal basis in these three spaces is given by $\{e_k\}_{k=1}^{\infty}$, where

$$e_k(x)=\sqrt{\frac{2}{\pi}}\,\sin(kx)\,,\quad \|e_k\|_2=1\,,\quad \|e_k\|_{H^1}=k\,,\quad \|e_k\|_{H^2}=k^2\,.$$

We denote by $H^*(0,\pi)$ the dual space of $H^2\cap H^1_0(0,\pi)$ and by $\langle\cdot,\cdot\rangle_*$ the duality pairing, whereas we denote by $H^{-1}(0,\pi)$ the dual space of $H^1_0(0,\pi)$ and by $\langle\cdot,\cdot\rangle_1$ the duality pairing.

Let ξ be as in (9); we also consider the Sturm-Liouville eigenvalue problem

$$-\left(\frac{H_0}{\xi(x)^2}\,u'\right)'=\lambda\xi(x)u\quad \text{in } (0,\pi)\,,\qquad u(0)=u(\pi)=0\,. \tag{37}$$

We say that λ is an eigenvalue of (37) if there exists $u\neq 0$ (eigenfunction) such that

$$\int_0^\pi \frac{H_0}{\xi(x)^2}\,u'v'=\lambda\int_0^\pi \xi(x)uv\qquad \forall v\in H^1_0(0,\pi)\,.$$

It is well-known (see e.g. [1, §2.4]) that (37) admits a sequence of positive eigenvalues $\{\lambda_k\}$ which diverges to infinity. Each eigenvalue has multiplicity 1 and the sequence of eigenfunctions $\{u_k\}$ is an orthogonal basis of $L^2(0,\pi)$ and of $H^1_0(0,\pi)$ endowed with the scalar products

$$(u,v)_\xi=\int_0^\pi \xi(x)\,uv\qquad \forall u,v\in L^2_\xi(0,\pi)\,,\qquad (u,v)_{H^1_\xi}=\int_0^\pi \frac{H_0}{\xi(x)^2}\,u'v'\qquad \forall u,v\in H^1_\xi(0,\pi)\,.$$

Let us emphasize that the corresponding norms $\|\cdot\|_\xi$ and $\|\cdot\|_{H^1_\xi}$ are equivalent, respectively, to the norms $\|\cdot\|_2$ and $\|\cdot\|_{H^1}$ but, for the sake of clarity, we maintain the different notations L^2_ξ and H^1_ξ. In the sequel, we assume that $\{u_k\}$ is normalized in L^2_ξ, that is,

$$\|u_k\|_\xi=1\,,\quad \|u_k\|_{H^1_\xi}=\sqrt{\lambda_k}\,.$$

We denote by $H^\xi(0,\pi)$ the dual space of $H^1_\xi(0,\pi)$ and by $\langle\cdot,\cdot\rangle_\xi$ the corresponding duality pairing.

We set

$$\Phi(s):=F(s)-\frac{Mg}{2}\qquad\text{and}\qquad \Psi(s)=\int_0^s\Phi(\sigma)\,d\sigma$$

so that

$$\Phi(0)=\Psi(0)=0\ ,\qquad \Phi(s)\geq 0\,,\ \Psi(s)\geq 0\quad \forall s\in\mathbb{R}\,.$$

In the above functional-analytic setting and with this simplification, the equations (31)-(32)-(33)-(34) may be rewritten as

$$\begin{cases} m\xi(x)\,\ddot p_1 &=\left[\frac{H_0}{\xi(x)^2}p_1'\right]'+\frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)p_1'(x)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3}+\Phi(y+\ell\theta-p_1)\\ m\xi(x)\,\ddot p_2 &=\left[\frac{H_0}{\xi(x)^2}p_2'\right]'+\frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)p_2'(x)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3}+\Phi(y-\ell\theta-p_2)\\ M\ddot y &=-EIy''''-\Phi(y+\ell\theta-p_1)-\Phi(y-\ell\theta-p_2)\\ \frac{M}{3}\ell^2\ddot\theta &=GK\theta''+\ell\Phi(y-\ell\theta-p_2)-\ell\Phi(y+\ell\theta-p_1)\end{cases} \tag{38}$$

where $\xi(x)$ and $s(x)$ are linked through (9). We add the boundary conditions (35)-(36) which we rewrite for $L=\pi$:

$$y(0,t)=y(\pi,t)=y''(0,t)=y''(\pi,t)=p_i(0,t)=p_i(\pi,t)=\theta(0,t)=\theta(\pi,t)=0\quad \forall t\geq 0\,. \tag{39}$$

We fix some initial data at time $t=0$,

$$
\begin{aligned}
&y(x,0)=y^0(x)\,, \quad p_i(x,0)=p_i^0(x)\,, \quad \theta(x,0)=\theta^0(x) \qquad \forall x\in(0,\pi)\\
&\dot y(x,0)=y^1(x)\,, \quad \dot p_i(x,0)=p_i^1(x)\,, \quad \dot\theta(x,0)=\theta^1(x) \qquad \forall x\in(0,\pi)\,,
\end{aligned}
\tag{40}
$$

with the following regularity

$$
y^0\in H^2\cap H^1_0(0,\pi)\,, \quad \theta^0,p_i^0\in H^1_0(0,\pi)\,, \quad y^1,\theta^1,p_i^1\in L^2(0,\pi)\,. \tag{41}
$$

We say that (p_1,p_2,y,θ) is a weak solution of (38) if

$$
\begin{aligned}
\theta,p_i&\in C^0\Big([0,T];H^1_0(0,\pi)\Big)\cap C^1\Big([0,T];L^2(0,\pi)\Big)\cap C^2\Big([0,T];H^{-1}(0,\pi)\Big)\\
y&\in C^0\Big([0,T];H^2\cap H^1_0(0,\pi)\Big)\cap C^1\Big([0,T];L^2(0,\pi)\Big)\cap C^2\Big([0,T];H^*(0,\pi)\Big)
\end{aligned}
\tag{42}
$$

and if the following equations are satisfied

$$
\left\{
\begin{array}{l}
m\langle\ddot p_1,v_1\rangle_\xi+(p_1,v_1)_{H^1_\xi}=\frac{AE}{L_c}\left[\int_0^\pi\frac{s'(x)p_1'(x)}{\xi(x)}dx\right]\left(\frac{s''(x)}{\xi(x)^3},v_1\right)_2+\Big(\Phi(y+\ell\theta-p_1),v_1\Big)_2\\
m\langle\ddot p_2,v_2\rangle_\xi+(p_2,v_2)_{H^1_\xi}=\frac{AE}{L_c}\left[\int_0^\pi\frac{s'(x)p_2'(x)}{\xi(x)}dx\right]\left(\frac{s''(x)}{\xi(x)^3},v_2\right)_2+\Big(\Phi(y-\ell\theta-p_2),v_2\Big)_2\\
M\langle\ddot y,w\rangle_*+EI(y,w)_{H^2}=-\Big(\Phi(y+\ell\theta-p_1)+\Phi(y-\ell\theta-p_2),w\Big)_2\\
\frac{M}{3}\ell^2\langle\ddot\theta,v_3\rangle_1+GK(\theta,v_3)_{H^1}=\Big(\ell\Phi(y-\ell\theta-p_2)-\ell\Phi(y+\ell\theta-p_1),v_3\Big)_2
\end{array}
\right.
$$

for all $v_i\in H^1_0(0,\pi)$, $w\in H^2\cap H^1_0(0,\pi)$ and $t>0$. In the sequel we denote by X_T the functional space of solutions:

$$
(p_1,p_2,y,\theta)\in X_T \iff \text{(42) holds.} \tag{43}
$$

Note that this space already includes the boundary conditions (39): hence, from now on we will not mention further (39). In Subsection 3.3 we prove the existence and uniqueness of a weak solution of (38), see Theorem 2. Under suitable regularity assumptions on the data one can then obtain the same result for a strong solution of (38), see [28].

In view of the energy balance performed in Section 2.3, the conserved (and approximated at second order) total energy of any solution $(p_1,p_2,y,\theta)\in X_T$ of (38) is given by

$$
\begin{aligned}
E(t) \;=\;& \int_0^\pi\left(\frac{M}{6}\ell^2\dot\theta^2+\frac{M}{2}\dot y^2+\frac{m}{2}(\dot p_1^2+\dot p_2^2)\,\xi(x)\right)dx+\int_0^\pi\left(\frac{EI}{2}(y'')^2+\frac{GK}{2}(\theta')^2\right)dx\\
&+\frac{AE}{2L_c}\left(\int_0^\pi\frac{s'\,p_1'}{\xi(x)}\,dx\right)^2+\frac{AE}{2L_c}\left(\int_0^\pi\frac{s'\,p_2'}{\xi(x)}\,dx\right)^2+H_0\int_0^\pi\left(\frac{(p_1')^2+(p_2')^2}{2\xi(x)^2}-s'\,(p_1'+p_2')\right)dx\\
&-mg\int_0^\pi(p_1+p_2)\xi(x)\,dx+\int_0^\pi\Big(\Psi(y+\ell\theta-p_1)+\Psi(y-\ell\theta-p_2)\Big)\,dx
\end{aligned}
\tag{44}
$$

where, in the first line we see the total kinetic energy of the bridge and the elastic energy of the deck whereas in the second line we see the stretching energy of the two cables. The third line in (44) deserves a particular attention: we see there the gravitational energies of the cables and the deck, the latter being included in Ψ while the former cancels in the equations (38) due to its presence with an opposite sign in both (24) and in the gravitational energy.

3.2 Existence and uniqueness for a related linear problem

We consider here the linear and decoupled problem

$$\begin{cases} m\xi(x)\,\ddot p_1 &= \left[\frac{H_0}{\xi(x)^2}p_1'\right]' + g_1(x,t) \\ m\xi(x)\,\ddot p_2 &= \left[\frac{H_0}{\xi(x)^2}p_2'\right]' + g_2(x,t) \\ M\ddot y &= -EIy'''' + g_3(x,t) \\ \frac{M}{3}\ell^2\ddot\theta &= GK\theta'' + g_4(x,t) \end{cases} \qquad x\in(0,\pi)\,,\ t>0\,. \tag{45}$$

where $g_j \in C^0([0,\pi]\times[0,T])$ for $j=1,...,4$. To (45) we associate the initial conditions (40) with the regularity as in (41). We say that $(p_1,p_2,y,\theta)\in X_T$ is a weak solution of (45) if

$$\begin{cases} m\langle \ddot p_1, v\rangle_\xi + (p_1,v)_{H^1_\xi} = (g_1,v)_2 \\ m\langle \ddot p_2, v\rangle_\xi + (p_2,v)_{H^1_\xi} = (g_2,v)_2 \\ M\langle \ddot y, w\rangle_* + EI(y,w)_{H^2} = (g_3,w)_2 \\ \frac{M}{3}\ell^2\langle\ddot\theta,v\rangle_1 + GK(\theta,v)_{H^1} = (g_4,v)_2 \end{cases} \qquad \forall v\in H^1_0(0,\pi)\,,\ \forall w\in H^2\cap H^1_0(0,\pi)\,,\ t>0\,.$$

The purpose of the present section is to establish the existence and uniqueness of a weak solution of (45).

Theorem 1. *Let $T>0$ and let $g_j\in C^0([0,\pi]\times[0,T])$ for $j=1,...,4$. For all $y^0,\theta^0,p_i^0,y^1,\theta^1,p_i^1$ satisfying* (41) *there exists a unique weak solution $(p_1,p_2,y,\theta)\in X_T$ of* (45) *satisfying* (40).

The proof of Theorem 1 uses a Galerkin procedure and is divided in several steps, see e.g. [28, Theorem II.4.1]. The proof allows to emphasize the delicate role of all the spaces, norms, and scalar products defined in Section 3.1. Moreover, the proof provides the underlying idea for a numerical approximation of the dynamics with a finite number of modes. For these reasons, and for the sake of completeness, we give here the full proof of Theorem 1.

Step 1. We construct a sequence of solutions of approximated problems in finite dimensional spaces.

We consider the two basis defined in Section 3.1 and, for any $n\geq 1$, we put

$$E_n := \mathrm{span}\{e_1,\dots,e_n\}\,,\qquad U_n := \mathrm{span}\{u_1,\dots,u_n\}\,.$$

For any $n\geq1$ we also put

$$(p_i^0)_n := \sum_{k=1}^n (p_i^0,u_k)_\xi\, u_k = -\sum_{k=1}^n \lambda_k^{-1}(p_i^0,u_k)_{H^1_\xi}\, u_k\,,\qquad (p_i^1)_n := \sum_{k=1}^n (p_i^1,u_k)_\xi\, u_k\,,$$

$$y_n^0 := \sum_{k=1}^n (y^0,e_k)_2\, e_k = \sum_{k=1}^n k^{-4}(y^0,e_k)_{H^2}\, e_k\,,\qquad y_n^1 := \sum_{k=1}^n (y^1,e_k)_2\, e_k\,,$$

$$\theta_n^0 := \sum_{k=1}^n (\theta^0,e_k)_2\, e_k = -\sum_{k=1}^n k^{-2}(\theta^0,e_k)_{H^1}\, e_k\,,\qquad \theta_n^1 := \sum_{k=1}^n (\theta^1,e_k)_2\, e_k\,,$$

so that

$$y_n^0\to y^0 \text{ in } H^2\,,\quad \theta_n^0\to\theta^0\,,\ (p_i^0)_n\to p_i^0 \text{ in } H^1\,,\quad y_n^1\to y^1\,,\ \theta_n^1\to\theta^1\,,\ (p_i^1)_n\to p_i^1 \text{ in } L^2 \tag{46}$$

as $n \to \infty$. For any $n \geq 1$ we seek $((p_1)_n, (p_2)_n, y_n, \theta_n)$ such that

$$(p_i)_n(x,t) = \sum_{k=1}^{n} (p_i)_n^k(t) u_k\,, \qquad y_n(x,t) = \sum_{k=1}^{n} y_n^k(t) e_k\,, \qquad \theta_n(x,t) = \sum_{k=1}^{n} \theta_n^k(t) e_k$$

and solving the variational problem

$$\begin{cases} m((\ddot{p}_1)_n, v)_\xi + ((p_1)_n, v)_{H^1_\xi} = (g_1, v)_2 \\ m((\ddot{p}_2)_n, v)_\xi + ((p_2)_n, v)_{H^1_\xi} = (g_2, v)_2 \\ M(\ddot{y}_n, w)_2 + EI(y_n, w)_{H^2} = (g_3, w)_2 \\ \frac{M}{3}\ell^2(\ddot{\theta}_n, w)_2 + GK(\theta_n, w)_{H^1} = (g_4, w)_2 \end{cases} \qquad \forall v \in U_n\,,\ \forall w \in E_n\,,\ t > 0\,. \tag{47}$$

By making n tests on each equation (for $v = u_1, ..., u_n$ and $w = e_1, ..., e_n$) we obtain the $4n$ linear equations

$$\begin{cases} m(\ddot{p}_1)_n^k + \lambda_k (p_1)_n^k = (g_1, u_k)_2 \\ m(\ddot{p}_2)_n^k + \lambda_k (p_2)_n^k = (g_2, u_k)_2 \\ M\ddot{y}_n^k + EI\, k^4\, y_n^k = (g_3, e_k)_2 \\ \frac{M}{3}\ell^2 \ddot{\theta}_n^k + GK\, k^2\, \theta_n^k = (g_4, e_k)_2 \end{cases} \qquad \forall k = 1, ..., n\,. \tag{48}$$

It is a classical result from the theory of linear ODE's that this finite-dimensional linear system, together with the initial conditions

$$(p_i)_n^k(0) = (p_i^0, u_k)_\xi\,,\ (\dot{p}_i)_n^k(0) = (p_i^1, u_k)_\xi\,,\ y_n^k(0) = (y^0, e_k)_2\,,\ \dot{y}_n^k(0) = (y^1, e_k)_2\,,$$
$$\theta_n^k(0) = (\theta^0, e_k)_2\,,\ \dot{\theta}_n^k(0) = (\theta^1, e_k)_2\,,$$

admits a unique global solution defined for all $t > 0$.

Step 2. In this step we prove a local uniform bound for the sequence $\{((p_1)_n, (p_2)_n, y_n, \theta_n)\}$.

We fix some (finite) $T > 0$; in what follows the c_i's denote positive constants independent of n, possibly depending on T. We take $w = \dot{y}_n$ in $(47)_3$ and we integrate first over $(0, \pi)$ and then over $(0, t)$ for some $t \in (0, T)$ to obtain

$$\begin{aligned} M\,\|\dot{y}_n(t)\|_2^2 + EI\,\|y_n(t)\|_{H^2}^2 \quad &= \quad c_1 + 2\int_0^t (g_3, \dot{y}_n)_2 \leq c_1 + 2\|g_3\|_\infty \int_0^t \|\dot{y}_n(\tau)\|_1\, d\tau \\ &\leq \quad c_1 + c_2 \int_0^t \|\dot{y}_n(\tau)\|_1\, d\tau \end{aligned} \tag{49}$$

where $\|\cdot\|_\infty$ denotes the $L^\infty([0,\pi] \times [0,T])$-norm whereas $\|\cdot\|_1$ denotes the $L^1(0,\pi)$-norm. Here, $c_1 := \sup_n (M\|y_n^1\|_2^2 + EI\|y_n^0\|_{H^2}^2) < \infty$. By using the Hölder inequality, we see that (49) implies

$$\|\dot{y}_n(t)\|_2^2 \leq c_3 + c_4 \left(\int_0^t \|\dot{y}_n(\tau)\|_2^2\, d\tau \right)^{1/2}. \tag{50}$$

In turn, (50) may be written as

$$f'(t) \leq c_3 + c_4\sqrt{f(t)} \qquad \forall t \in (0,T)\,, \qquad f(t) := \int_0^t \|\dot{y}_n(\tau)\|_2^2\, d\tau\,.$$

We now recall a Gronwall-type lemma which can be deduced (e.g.) from [4] and [8, Lemma A.5/p.157].

Lemma 1. *Let $f \in C^1(\mathbb{R}_+)$ be such that $f(0) = 0$, $0 \le f'(t) \le C_1 + C_2 \sqrt{f(t)}$ for all $t \ge 0$ (for some $C_1, C_2 > 0$). Then*

$$f(t) \le \frac{(C_1 + C_2)^2}{4} t^2 + (C_1 + C_2)\, t \qquad \forall t \ge 0\,.$$

By applying Lemma 1 to (50) and going back to (49), we obtain constants $c_5, c_6 > 0$ (independent of n) such that

$$\|\dot{y}_n(t)\|_2^2 + \|y_n(t)\|_{H^2}^2 \le c_5 + c_6\, t^2 \qquad \forall t \in (0, T)\ . \tag{51}$$

Similarly, take $v = (\dot{p}_1)_n$ in $(47)_1$ or $v = (\dot{p}_2)_n$ in $(47)_2$, to obtain constants $c_7, c_8, c_9, c_{10} > 0$ (independent of n) such that

$$\|(\dot{p}_1)_n(t)\|_\xi^2 + \|(p_1)_n(t)\|_{H_\xi^1}^2 \le c_7 + c_8\, t^2 \qquad \forall t \in (0, T)\ ,$$

$$\|(\dot{p}_2)_n(t)\|_\xi^2 + \|(p_2)_n(t)\|_{H_\xi^1}^2 \le c_9 + c_{10}\, t^2 \qquad \forall t \in (0, T)\ .$$

In order to obtain these inequalities one also needs to combine the Hölder inequality with the equivalence of the norms in L^2 and L^2_ξ.

Finally, we take $w = \dot{\theta}_n$ in $(47)_4$. Proceeding as above, we obtain constants $c_{11}, c_{12} > 0$ (independent of n) such that

$$\|\dot{\theta}_n(t)\|_2^2 + \|\theta_n(t)\|_{H^1}^2 \le c_{11} + c_{12}\, t^2 \qquad \forall t \in (0, T)\ .$$

Step 3. We show that $\{((p_1)_n, (p_2)_n, y_n, \theta_n)\}$ admits a strongly convergent subsequence in some norm.

Let us consider in detail the sequence $\{y_n\}$, the other components being similar. The bound in (51) suggests to prove strong convergence in the space

$$C^0([0, T]; H^2_*(0, \pi)) \cap C^1([0, T]; L^2(0, \pi))\,.$$

The equations in (48) show that the components y_n^k do not depend on n, that is,

$$y_n(x, t) = \sum_{k=1}^{n} y^k(t) e_k\,.$$

Take some $m > n \ge 1$ and define

$$y_{m,n}(x, t) := y_m(x, t) - y_n(x, t) = \sum_{k=n+1}^{m} y^k(t) e_k$$

so that, in particular,

$$y_{m,n}(x, 0) = y_m^0 - y_n^0\,, \quad \dot{y}_{m,n}(x, 0) = y_m^1 - y_n^1\,.$$

Consider also the Fourier decomposition of the function g_3:

$$g_3(x, t) = \sum_{k=1}^{\infty} g_3^k(t) e_k\,;$$

since $g_3 \in C^0([0, \pi] \times [0, T]) \subset L^2((0, \pi) \times (0, T))$, we know that

$$\left(\sum_{k=n+1}^{\infty} g_3^k(t)^2 \right)^{1/2} \to 0 \quad \text{as } n \to \infty \qquad \text{in } L^2(0, T)\,. \tag{52}$$

Rewrite $(47)_3$ with n replaced by m. Then by taking $w = \dot{y}_{m,n}(t)$ as a test function, by using the orthogonality of the system $\{e_k\}$, and by integrating over $(0,t)$ we obtain

$$\begin{aligned} M\|\dot{y}_{m,n}(t)\|_2^2 + EI\|y_{m,n}(t)\|_{H^2}^2 &= C_{m,n} + 2\int_0^t (g_3, \dot{y}_{m,n})_2 = C_{m,n} + 2\int_0^t \left(\sum_{k=n+1}^m g^k(\tau) e_k, \dot{y}_{m,n} \right)_2 d\tau \\ &\le C_{m,n} + 2\int_0^T \left(\sum_{k=n+1}^m g_3^k(t)^2 \right)^{1/2} \|\dot{y}_{m,n}(t)\|_2 dt \end{aligned} \tag{53}$$

where $C_{m,n} = M\|y_m^1 - y_n^1\|_2^2 + EI\|y_m^0 - y_n^0\|_{H^2}^2$. By (46) we know that $C_{m,n} \to 0$ as $m, n \to \infty$; combined with (52) and (53), this shows that $\{y_n\}$ is a Cauchy sequence in $C^0([0,T]; H^2 \cap H_0^1(0,\pi)) \cap C^1([0,T]; L^2(0,\pi))$. By completeness of these spaces we conclude that

$$\begin{aligned} &\exists y \in C^0([0,T]; H^2 \cap H_0^1(0,\pi)) \cap C^1([0,T]; L^2(0,\pi)) \text{ s.t.} \\ &y_n \to y \quad \text{in } C^0([0,T]; H^2 \cap H_0^1(0,\pi)) \cap C^1([0,T]; L^2(0,\pi)) \quad \text{as } n \to +\infty\,, \end{aligned} \tag{54}$$

thereby completing the proof of the claim.

For the sequences $\{\theta_n\}$, $\{(p_1)_n\}$, $\{(p_2)_n\}$ we may proceed similarly and prove that

$$\begin{aligned} &\exists p_i \in C^0\Big([0,T]; H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big) \text{ s.t.} \\ &(p_i)_n \to p_i \quad \text{in } C^0\Big([0,T]; H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big) \quad \text{as } n \to +\infty\,, \end{aligned} \tag{55}$$

$$\begin{aligned} &\exists \theta \in C^0\Big([0,T]; H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big) \text{ s.t.} \\ &\theta_n \to \theta \quad \text{in } C^0\Big([0,T]; H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big) \quad \text{as } n \to +\infty\,. \end{aligned} \tag{56}$$

Step 4. We take the limit in (47) and we prove Theorem 1.

By using (54)-(55)-(56) in (47) we see that (p_1, p_2, y, θ) is a weak solution of (45) satisfying (40), with the additional regularity

$$\theta, p_i \in C^2\Big([0,T]; H^{-1}(0,\pi)\Big)\,, \quad y \in C^2\Big([0,T]; H^*(0,\pi)\Big)$$

following from the equations (45). Therefore, for any $T > 0$ we have proved the existence of a weak solution $(p_1, p_2, y, \theta) \in X_T$ of (45) over the interval $(0,T)$ and satisfying (40). By arbitrariness of $T > 0$ this proves global existence. Finally, arguing by contradiction and assuming the existence of two solutions, we subtract the two linear equations for the two solutions and we obtain an homogeneous linear problem; the energy conservation then shows that the (nonnegative) energy is always 0, which proves that the two solutions are, in fact, the same. This completes the proof of Theorem 1.

3.3 The existence and uniqueness result

The purpose of this final subsection is to prove the following statement.

Theorem 2. *For all $y^0, \theta^0, p_i^0, y^1, \theta^1, p_i^1$ satisfying* (41) *there exists a unique (global in time) weak solution $(p_1, p_2, y, \theta) \in X_\infty$ of* (38) *satisfying the initial conditions* (40).

Global in time means here that we can take any $T > 0$, including $T = \infty$; this explains the notation X_∞. For the proof we use a fixed point procedure combined with an energy estimate. The first step consists in defining the map which will be shown to have a fixed point; in the sequel we emphasize the dependence on time of the nonlocal term.

Lemma 2. *Let $T > 0$ and assume that $y^0, \theta^0, p_i^0, y^1, \theta^1, p_i^1$ satisfy* (41). *For all $(q_1, q_2, z, \alpha) \in X_T$ there exists a unique weak solution $(p_1, p_2, y, \theta) \in X_T$ of the system*

$$\begin{aligned}
m\xi(x)\,\ddot{p}_1 &= \left[\frac{H_0}{\xi(x)^2}p_1'\right]' + \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)q_1'(x,t)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} + \Phi(z+\ell\alpha-q_1)\\
m\xi(x)\,\ddot{p}_2 &= \left[\frac{H_0}{\xi(x)^2}p_2'\right]' + \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)q_2'(x,t)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} + \Phi(z-\ell\alpha-q_2)\\
M\ddot{y} &= -EIy'''' - \Phi(z+\ell\alpha-q_1) - \Phi(z-\ell\alpha-q_2)\\
\frac{M}{3}\ell^2\ddot{\theta} &= GK\theta'' + \ell\Phi(z-\ell\alpha-q_2) - \ell\Phi(z+\ell\alpha-q_1)
\end{aligned}$$

satisfying the initial conditions (40).

Proof. We set

$$\begin{aligned}
g_1(x,t) &= \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)q_1'(x,t)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} + \Phi(z+\ell\alpha-q_1)\,,\\
g_2(x,t) &= \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)q_2'(x,t)}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} + \Phi(z-\ell\alpha-q_2)\,,\\
g_3(x,t) &= -\Phi(z+\ell\alpha-q_1) - \Phi(z-\ell\alpha-q_2)\,,\\
g_4(x,t) &= \ell\Phi(z-\ell\alpha-q_2) - \ell\Phi(z+\ell\alpha-q_1)\,.
\end{aligned}$$

Then $g_j \in C^0([0,\pi]\times[0,T])$ for $j = 1, ..., 4$ and we apply Theorem 1. □

Denote by Z_T the space of functions (p_1, p_2, y, θ) such that

$$\begin{aligned}
\theta, p_i &\in C^0\Big([0,T]; H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big)\\
y &\in C^0\Big([0,T]; H^2 \cap H_0^1(0,\pi)\Big) \cap C^1\Big([0,T]; L^2(0,\pi)\Big)
\end{aligned} \tag{57}$$

which is a Banach space when endowed with the norm

$$\begin{aligned}
\|(p_1,p_2,y,\theta)\|_{Z_T}^2 \;:=\; & \|p_1\|^2_{L^\infty(H^1_\xi)} + \|\dot{p}_1\|^2_{L^\infty(L^2_\xi)} + \|p_2\|^2_{L^\infty(H^1_\xi)} + \|\dot{p}_2\|^2_{L^\infty(L^2_\xi)}\\
& + \|y\|^2_{L^\infty(H^2)} + \|\dot{y}\|^2_{L^\infty(L^2)} + \|\theta\|^2_{L^\infty(H^1)} + \|\dot{\theta}\|^2_{L^\infty(L^2)}
\end{aligned}$$

where the norms $L^\infty(\cdot)$ have the usual time-space meaning. Clearly, X_T is a subspace of Z_T, see (43). For any $y^0, \theta^0, p_i^0, y^1, \theta^1, p_i^1$ satisfying (41) we define the convex subset (complete metric space)

$$\mathbf{B} := \{(p_1, p_2, y, \theta) \in Z_T;\ (40) \text{ holds}\}\,.$$

If we take (q_1, q_2, z, α) and (p_1, p_2, y, θ) both satisfying (40), then Lemma 2 defines a map

$$\Upsilon : \mathbf{B} \to \mathbf{B}\,, \qquad \Upsilon(q_1, q_2, z, \alpha) = (p_1, p_2, y, \theta)\,. \tag{58}$$

We now prove that for sufficiently small T this map is contractive.

Lemma 3. *Let $y^0, \theta^0, p_i^0, y^1, \theta^1, p_i^1$ satisfy* (41). *If $T > 0$ is sufficiently small, then the map Υ defined in* (58) *is a contraction from $\mathbf{B}$ into $\mathbf{B}$.*

Proof. Consider two different $(q_1^1, q_2^1, z^1, \alpha^1)$ and $(q_1^2, q_2^2, z^2, \alpha^2)$ in $\mathbf{B}$ and let $(p_1^j, p_2^j, y^j, \theta^j) = \Upsilon(q_1^j, q_2^j, z^j, \alpha^j)$ for $j = 1, 2$. Denote by

$$p_1 = p_1^1 - p_1^2\,, \quad p_2 = p_2^1 - p_2^2\,, \quad y = y^1 - y^2\,, \quad \theta = \theta^1 - \theta^2\,.$$

We underline that these notations should not be confused with the initial conditions (40). By subtracting the two systems satisfied by $(p_1^j, p_2^j, y^j, \theta^j)$ we see that (p_1, p_2, y, θ) is a weak solution of

$$\begin{aligned} m\xi(x)\,\ddot{p}_1 &= \left[\frac{H_0}{\xi(x)^2}p_1'\right]' + \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)[(q_1^1)'(x,t) - (q_1^2)'(x,t)]}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} \\ &\quad +\Phi(z^1 + \ell\alpha^1 - q_1^1) - \Phi(z^2 + \ell\alpha^2 - q_1^2) \end{aligned} \tag{59}$$

$$\begin{aligned} m\xi(x)\,\ddot{p}_2 &= \left[\frac{H_0}{\xi(x)^2}p_2'\right]' + \frac{AE}{L_c}\left[\int_0^\pi \frac{s'(x)[(q_2^1)'(x,t) - (q_2^2)'(x,t)]}{\xi(x)}dx\right]\frac{s''(x)}{\xi(x)^3} \\ &\quad +\Phi(z^1 - \ell\alpha^1 - q_2^1) - \Phi(z^2 - \ell\alpha^2 - q_2^2) \end{aligned} \tag{60}$$

$$\begin{aligned} M\ddot{y} &= -EIy'''' - \Phi(z^1 + \ell\alpha^1 - q_1^1) - \Phi(z^1 - \ell\alpha^1 - q_2^1) \\ &\quad +\Phi(z^2 + \ell\alpha^2 - q_1^2) + \Phi(z^2 - \ell\alpha^2 - q_2^2) \end{aligned} \tag{61}$$

$$\begin{aligned} \frac{M}{3}\ell^2\ddot{\theta} &= GK\theta'' + \ell[\Phi(z^1 - \ell\alpha^1 - q_2^1) - \Phi(z^1 + \ell\alpha^1 - q_1^1)] \\ &\quad -\ell[\Phi(z^2 - \ell\alpha^2 - q_2^2) - \Phi(z^2 + \ell\alpha^2 - q_1^2)] \end{aligned} \tag{62}$$

satisfying homogeneous initial conditions.

Multiply (59) by $\dot{p}_1$, (60) by $\dot{p}_2$, (61) by $\dot{y}$, (62) by $\dot{\theta}$. Then integrate with respect to x over $(0, \pi)$ and with respect to t over $(0, t)$; we obtain

$$\begin{aligned} m\|\dot{p}_1(t)\|_\xi^2 + \|p_1(t)\|_{H_\xi^1}^2 &= 2\frac{AE}{L_c}\int_0^t \left[\int_0^\pi \frac{s'(x)[(q_1^1)'(x,\tau) - (q_1^2)'(x,\tau)]}{\xi(x)}dx\right]\left[\int_0^\pi \frac{s''(x)}{\xi(x)^3}\dot{p}_1 dx\right]d\tau \\ &\quad +2\int_0^t\int_0^\pi [\Phi(z^1+\ell\alpha^1-q_1^1) - \Phi(z^2+\ell\alpha^2-q_1^2)]\dot{p}_1 dx d\tau \end{aligned} \tag{63}$$

$$\begin{aligned} m\|\dot{p}_2(t)\|_\xi^2 + \|p_2(t)\|_{H_\xi^1}^2 &= 2\frac{AE}{L_c}\int_0^t \left[\int_0^\pi \frac{s'(x)[(q_2^1)'(x,\tau) - (q_2^2)'(x,\tau)]}{\xi(x)}dx\right]\left[\int_0^\pi \frac{s''(x)}{\xi(x)^3}\dot{p}_2 dx\right]d\tau \\ &\quad +2\int_0^t\int_0^\pi [\Phi(z^1-\ell\alpha^1-q_2^1) - \Phi(z^2-\ell\alpha^2-q_2^2)]\dot{p}_2 dx d\tau \end{aligned} \tag{64}$$

$$\begin{aligned} M\|\dot{y}(t)\|_2^2 + EI\|y(t)\|_{H^2}^2 &= 2\int_0^t\int_0^\pi [\Phi(z^2+\ell\alpha^2-q_1^2) - \Phi(z^1+\ell\alpha^1-q_1^1)]\dot{y} dx d\tau \\ &\quad +2\int_0^t\int_0^\pi [\Phi(z^2-\ell\alpha^2-q_2^2) - \Phi(z^1-\ell\alpha^1-q_2^1)]\dot{y} dx d\tau \end{aligned} \tag{65}$$

$$\begin{aligned} \frac{M\ell^2}{3}\|\dot{\theta}(t)\|_2^2 + GK\|\theta(t)\|_{H^1}^2 &= 2\ell\int_0^t\int_0^\pi [\Phi(z^1-\ell\alpha^1-q_2^1) - \Phi(z^2-\ell\alpha^2-q_2^2)]\dot{\theta} dx d\tau \\ &\quad -2\ell\int_0^t\int_0^\pi [\Phi(z^1+\ell\alpha^1-q_1^1) - \Phi(z^2+\ell\alpha^2-q_1^2)]\dot{\theta} dx d\tau. \end{aligned} \tag{66}$$

Our purpose is to add all the above identities and to obtain estimates. By adding the four left hand sides of (63)-(64)-(65)-(66), we find $\delta > 0$ such that

$$m\|\dot{p}_1(t)\|_\xi^2 + \|p_1(t)\|_{H_\xi^1}^2 + m\|\dot{p}_2(t)\|_\xi^2 + \|p_2(t)\|_{H_\xi^1}^2 + M\,\|\dot{y}(t)\|_2^2 + EI\,\|y(t)\|_{H^2}^2 + \frac{M}{3}\ell^2\|\dot{\theta}(t)\|_2^2 + GK\|\theta(t)\|_{H^1}^2$$

$$\geq \delta\Big(\|\dot p_1(t)\|_\xi^2+\|p_1(t)\|_{H^1_\xi}^2+\|\dot p_2(t)\|_\xi^2+\|p_2(t)\|_{H^1_\xi}^2+\|\dot y(t)\|_2^2+\|y(t)\|_{H^2}^2+\|\dot\theta(t)\|_2^2+\|\theta(t)\|_{H^1}^2\Big)\,. \quad (67)$$

Next, we seek an upper bound for the sum of the four right hand sides. Two kinds of terms appear: the Φ-terms and the nonlocal terms. For the term containing Φ in (63) we remark that

$$|\Phi(z^1+\ell\alpha^1-q_1^1)-\Phi(z^2+\ell\alpha^2-q_1^2)|\leq \kappa(x)\Big(|z^1-z^2|+\ell|\alpha^1-\alpha^2|+|q_1^1-q_1^2|\Big)\,;$$

therefore, by using the Hölder inequality several times,

$$\begin{aligned}
&\left|\int_0^t\!\!\int_0^\pi [\Phi(z^1+\ell\alpha^1-q_1^1)-\Phi(z^2+\ell\alpha^2-q_1^2)]\dot p_1\,dx d\tau\right| \\
&\leq c\int_0^t\!\!\int_0^\pi \Big(|z^1-z^2|+\ell|\alpha^1-\alpha^2|+|q_1^1-q_1^2|\Big)|\dot p_1|\,dx d\tau \\
&\leq c\int_0^t \Big(\|z^1-z^2\|_2+\|\alpha^1-\alpha^2\|_2+\|q_1^1-q_1^2\|_2\Big)\|\dot p_1(\tau)\|_\xi d\tau\,. \qquad (68)
\end{aligned}$$

Similarly, the terms containing Φ in (64)-(65)-(66) may be estimated, respectively, as follows

$$\Phi\text{-terms in (64)} \ \leq c\int_0^t \Big(\|z^1-z^2\|_2+\|\alpha^1-\alpha^2\|_2+\|q_2^1-q_2^2\|_2\Big)\|\dot p_2(\tau)\|_\xi d\tau\,, \qquad (69)$$

$$\Phi\text{-terms in (65)} \ \leq c\int_0^t \Big(\|z^1-z^2\|_2+\|\alpha^1-\alpha^2\|_2+\|q_1^1-q_1^2\|_2+\|q_2^1-q_2^2\|_2\Big)\|\dot y(\tau)\|_2 d\tau\,, \qquad (70)$$

$$\Phi\text{-terms in (66)} \ \leq c\int_0^t \Big(\|z^1-z^2\|_2+\|\alpha^1-\alpha^2\|_2+\|q_1^1-q_1^2\|_2+\|q_2^1-q_2^2\|_2\Big)\|\dot\theta(\tau)\|_2 d\tau\,. \qquad (71)$$

Then, we upper estimate the two nonlocal terms in (63)-(64):

$$\begin{aligned}
&\left|\int_0^t\left[\int_0^\pi \frac{s'(x)[(q_i^1)'(x,\tau)-(q_i^2)'(x,\tau)]}{\xi(x)}dx\right]\left[\int_0^\pi \frac{s''(x)}{\xi(x)^3}\dot p_i dx\right]d\tau\right| \\
&\leq c\int_0^t\left[\int_0^\pi |(q_i^1)'(x,\tau)-(q_i^2)'(x,\tau)|dx\right]\left[\int_0^\pi |\dot p_i|dx\right]d\tau \\
&\leq c\int_0^t \|(q_i^1)'(\tau)-(q_i^2)'(\tau)\|_2\|\dot p_i(\tau)\|_\xi d\tau\,. \qquad (72)
\end{aligned}$$

All together, (68)-(69)-(70)-(71)-(72) prove that

the sum of all the r.h.s. of (63)-(64)-(65)-(66)

$$\begin{aligned}
&\leq c\int_0^t\Big(\|z^1-z^2\|_2+\|\alpha^1-\alpha^2\|_2+\|q_1^1-q_1^2\|_2+\|q_2^1-q_2^2\|_2\Big)\Big(\|\dot p_1(\tau)\|_\xi+\|\dot p_2(\tau)\|_\xi+\|\dot y(\tau)\|_2+\|\dot\theta(\tau)\|_2\Big)d\tau \\
&\quad +c\int_0^t\Big(\|(q_1^1)'(\tau)-(q_1^2)'(\tau)\|_2+\|(q_2^1)'(\tau)-(q_2^2)'(\tau)\|_2\Big)\Big(\|\dot p_1(\tau)\|_\xi+\|\dot p_2(\tau)\|_\xi\Big)d\tau \\
&\leq c\sqrt{t}\Big(\|z^1-z^2\|_{L^\infty(L^2)}+\|\alpha^1-\alpha^2\|_{L^\infty(L^2)}+\|q_1^1-q_1^2\|_{L^\infty(L^2)}+\|q_2^1-q_2^2\|_{L^\infty(L^2)}\Big)\left(\int_0^t\big(\|\dot p_1(\tau)\|_\xi^2+\|\dot p_2(\tau)\|_\xi^2+\|\dot y(\tau)\|_2^2+\|\dot\theta(\tau)\|_2^2\big)d\tau\right)^{1/2} \\
&\quad +c\sqrt{t}\Big(\|(q_1^1)'-(q_1^2)'\|_{L^\infty(L^2)}+\|(q_2^1)'-(q_2^2)'\|_{L^\infty(L^2)}\Big)\left(\int_0^t\Big(\|\dot p_1(\tau)\|_\xi^2+\|\dot p_2(\tau)\|_\xi^2\Big)d\tau\right)^{1/2} \\
&\leq c\sqrt{T}\|(q_1^1-q_1^2,q_2^1-q_2^2,z^1-z^2,\alpha^1-\alpha^2)\|_{Z_T}\left(\int_0^t\big(\|\dot p_1(\tau)\|_\xi^2+\|\dot p_2(\tau)\|_\xi^2+\|\dot y(\tau)\|_2^2+\|\dot\theta(\tau)\|_2^2\big)d\tau\right)^{1/2}\,. \qquad (73)
\end{aligned}$$

By taking the sum of (63)-(64)-(65)-(66), by taking into account the lower bound (67) and the upper bound (73), for all $t \in (0,T)$ we obtain

$$\|\dot p_1(t)\|_\xi^2+\|p_1(t)\|_{H^1_\xi}^2+\|\dot p_2(t)\|_\xi^2+\|p_2(t)\|_{H^1_\xi}^2+\|\dot y(t)\|_2^2+\|y(t)\|_{H^2}^2+\|\dot\theta(t)\|_2^2+\|\theta(t)\|_{H^1}^2$$

$$\leq c\sqrt{T}\|(q_1^1-q_1^2,q_2^1-q_2^2,z^1-z^2,\alpha^1-\alpha^2)\|_{Z_T}\left(\int_0^t\big(\|\dot p_1(\tau)\|_\xi^2+\|\dot p_2(\tau)\|_\xi^2+\|\dot y(\tau)\|_2^2+\|\dot\theta(\tau)\|_2^2\big)d\tau\right)^{1/2}. \qquad (74)$$

In particular, by dropping the potential part, (74) implies that

$$\phi'(t)\leq K\sqrt{\phi(t)}\quad\text{with}\quad \phi(t)=\int_0^t\big(\|\dot p_1(\tau)\|_\xi^2+\|\dot p_2(\tau)\|_\xi^2+\|\dot y(\tau)\|_2^2+\|\dot\theta(\tau)\|_2^2\big)d\tau$$

$$\text{and}\quad K=c\sqrt{T}\|(q_1^1-q_1^2,q_2^1-q_2^2,z^1-z^2,\alpha^1-\alpha^2)\|_{Z_T}\,.$$

This differential inequality yields $\sqrt{\phi(t)}\leq Kt/2$ for all $t\in[0,T]$ which, inserted into the right hand side of (74), gives

$$\|\dot p_1(t)\|_\xi^2+\|p_1(t)\|_{H^1_\xi}^2+\|\dot p_2(t)\|_\xi^2+\|p_2(t)\|_{H^1_\xi}^2+\|\dot y(t)\|_2^2+\|y(t)\|_{H^2}^2+\|\dot\theta(t)\|_2^2+\|\theta(t)\|_{H^1}^2$$

$$\leq cT^2\|(q_1^1-q_1^2,q_2^1-q_2^2,z^1-z^2,\alpha^1-\alpha^2)\|_{Z_T}^2\qquad\forall t\in[0,T]\,.$$

By taking the supremum with respect to $t\in[0,T]$ and by replacing p_i, y, θ, this finally yields

$$\|(p_1^1-p_1^2,p_2^1-p_2^2,y^1-y^2,\theta^1-\theta^2)\|_{Z_T}\leq C\,T\,\|(q_1^1-q_1^2,q_2^1-q_2^2,z^1-z^2,\alpha^1-\alpha^2)\|_{Z_T}\,.$$

Then, for sufficiently small T the map Υ is contractive. □

Since a solution of (38) is a fixed point for Υ (as defined in (58)), Lemma 3 shows that there exists a unique weak solution $(p_1,p_2,y,\theta)\in X_T$ of (38) satisfying the initial conditions (40), provided that $T>0$ is sufficiently small. The conservation of the energy in (44) shows that the solution cannot blow up in finite time and therefore the solution is global, that is, $T=\infty$. This completes the proof of Theorem 2.

References

[1] M. Al-Gwaiz, *Sturm-Liouville theory and its applications*, Springer Undergraduate Mathematics Series, Springer-Verlag (2008)

[2] G. Arioli, F. Gazzola, *A new mathematical explanation of what triggered the catastrophic torsional mode of the Tacoma Narrows Bridge collapse*, Appl. Math. Modelling **39**, 901-912 (2015)

[3] G. Arioli, F. Gazzola, in preparation

[4] C. Baiocchi, *Soluzioni ordinarie e generalizzate del problema di Cauchy per equazioni differenziali astratte non lineari del secondo ordine in spazi di Hilbert*, Ricerche Mat. **16**, 27-95 (1967)

[5] E. Berchio, F. Gazzola, *A qualitative explanation of the origin of torsional instability in suspension bridges*, to appear in Nonlinear Analysis T.M.A.

[6] K.Y. Billah, R.H. Scanlan, *Resonance, Tacoma Narrows Bridge failure, and undergraduate physics textbooks*, Amer. J. Physics **59**, 118-124 (1991)

[7] M.A. Biot, T. von Kármán, *Mathematical methods in engineering: an introduction to the mathematical treatment of engineering problems*, McGraw Hill, New York (1940)

[8] H. Brezis, *Operateurs maximaux monotones*, North-Holland, Amsterdam (1973)

[9] J.M.W. Brownjohn, *Observations on non-linear dynamic characteristics of suspension bridges*, Earthquake Engineering & Structural Dynamics **23**, 1351-1367 (1994)

[10] P. Drábek, G. Holubová, A. Matas, P. Nečesal, *Nonlinear models of suspension bridges: discussion of the results*, Appl. Math. **48**, 497-514 (2003)

[11] F. Gazzola, *Mathematical models for suspension bridges*, MS&A Vol. 15, Springer (2015)

[12] F. Gazzola, H.-Ch. Grunau, G. Sweers, *Polyharmonic boundary value problems*, LNM **1991**, Springer (2010)

[13] F. Gazzola, M. Jleli, B. Samet, *On the Melan equation for suspension bridges*, to appear in J. Fixed Point Theory Appl.

[14] G. Holubová, A. Matas, *Initial-boundary value problem for nonlinear string-beam system*, J. Math. Anal. Appl. **288**, 784-802 (2003)

[15] H.M. Irvine, *Cable structures*, MIT Press Series in Structural Mechanics, Massachusetts (1981)

[16] A. Matas, J. Očenášek, *Modelling of suspension bridges*, Proceedings of Computational Mechanics **2**, 275-278 (2002)

[17] P.J. McKenna, *Torsional oscillations in suspension bridges revisited: fixing an old approximation*, Amer. Math. Monthly **106**, 1-18 (1999)

[18] P.J. McKenna, *Oscillations in suspension bridges, vertical and torsional*, Disc. Cont. Dynam. System S **7**, 785-791 (2014)

[19] P.J. McKenna, C.Ó Tuama, *Large torsional oscillations in suspension bridges visited again: vertical forcing creates torsional response*, Amer. Math. Monthly **108**, 738-745 (2001)

[20] P.J. McKenna, W. Walter, *Nonlinear oscillations in a suspension bridge*, Arch. Rat. Mech. Anal. **98**, 167-177 (1987)

[21] J. Melan, *Theory of arches and suspension bridges*, Myron Clark Publ. Comp. London (1913)

[22] K.S. Moore, *Large torsional oscillations in a suspension bridge: multiple periodic solutions to a nonlinear wave equation*, SIAM J. Math. Anal. **33**, 1411-1429 (2002)

[23] C.L. Navier, *Mémoire sur les ponts suspendus*, Imprimerie Royale, Paris (1823)

[24] W. Podolny, *Cable-suspended bridges*, In: Structural Steel Designers Handbook: AISC, AASHTO, AISI, ASTM, AREMA, and ASCE-07 Design Standards. By R.L. Brockenbrough and F.S. Merritt, 5^{th} Edition, McGraw-Hill, New York (2011)

[25] R.H. Scanlan, *The action of flexible bridges under wind, I: flutter theory, II: buffeting theory*, J. Sound and Vibration **60**, 187-199 & 201-211 (1978)

[26] F.C. Smith, G.S. Vincent, *Aerodynamic stability of suspension bridges: with special reference to the Tacoma Narrows Bridge, Part II: Mathematical analysis*, Investigation conducted by the Structural Research Laboratory, University of Washington - Seattle: University of Washington Press (1950)

[27] Tacoma Narrows Bridge collapse, http://www.youtube.com/watch?v=3mclp9QmCGs (1940)

[28] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, 68, Springer (1997)

[29] S.P. Timoshenko, *Theory of suspension bridges - Part I*, Journal of the Franklin Institute **235**, 213-238 (1943)

[30] S.P. Timoshenko, D.H. Young, *Theory of structures*, McGraw-Hill Kogakusha, Tokyo (1965)

[31] S. Woinowsky-Krieger, *The effect of an axial force on the vibration of hinged bars*, J. Appl. Mech. **17**, 35-36 (1950)